Further Math P1
Topical
2012-2021

Compiled By A Fellow Student

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Assalam-o-Alaikum,
So basically while doing AS I was unable to find any topical for FM that was not behind a paywall and easily accessible and due to that I was unable to practice much for the paper. Therefore, after my P1 paper I felt like I should compile this workbook so that others may have an easier time practicing and exploring this interesting/unique/difficult/depression inducing subject.

Oh and as only a student compiled this workbook there might be some errors. I have tried my best to eliminate them but sorry if you find any.

A fellow student,
Craftear/AN

ROOTS OF POLYNOMIAL

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Q1

3 The equation $x^4 - 2x^3 - 1 = 0$ has roots α, β, γ, δ.

(a) Find a quartic equation whose roots are α^3, β^3, γ^3, δ^3 and state the value of $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$. [4]

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(b) Find the value of $\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\frac{1}{\delta^3}$. [3]

(c) Find the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$. [2]

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Q2

2 The cubic equation $2x^3 - 4x^2 + 3 = 0$ has roots α, β, γ. Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(a) State the value of S_1 and find the value of S_2. [3]

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(b) **(i)** Express S_{n+3} in terms of S_{n+2} and S_n. [1]

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(ii) Hence, or otherwise, find the value of S_4. [2]

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(c) Use the substitution $y = S_1 - x$, where S_1 is the numerical value found in part **(a)**, to find and simplify an equation whose roots are $\alpha+\beta$, $\beta+\gamma$, $\gamma+\alpha$. [3]

(d) Find the value of $\frac{1}{\alpha+\beta}+\frac{1}{\beta+\gamma}+\frac{1}{\gamma+\alpha}$. [2]

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Q3

1 It is given that

$$\alpha+\beta+\gamma=3, \quad \alpha^2+\beta^2+\gamma^2=5, \quad \alpha^3+\beta^3+\gamma^3=6.$$

The cubic equation $x^3+bx^2+cx+d=0$ has roots α, β, γ.

Find the values of b, c and d. [6]

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Q4

4 The cubic equation $x^3 + 2x^2 + 3x + 3 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

(b) Show that $\alpha^3 + \beta^3 + \gamma^3 = 1$. [2]

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Q5

3 The cubic equation $x^3 + cx + 1 = 0$, where c is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^3, β^3, γ^3. [3]

(b) Show that $\alpha^6 + \beta^6 + \gamma^6 = 3 - 2c^3$. [3]

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(c) Find the real value of c for which the matrix $\begin{pmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{pmatrix}$ is singular. [5]

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Q6

1 The cubic equation $x^3 + bx^2 + cx + d = 0$, where b, c and d are constants, has roots α, β, γ. It is given that $\alpha\beta\gamma = -1$.

(a) State the value of d. [1]

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(b) Find a cubic equation, with coefficients in terms of b and c, whose roots are $\alpha + 1$, $\beta + 1$, $\gamma + 1$. [3]

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(c) Given also that $\gamma + 1 = -\alpha - 1$, deduce that $(c - 2b + 3)(b - 3) = b - c$. [4]

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Q7

2 The cubic equation $6x^3 + px^2 - 3x - 5 = 0$, where p is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^2, β^2, γ^2. [3]

(b) It is given that $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$.

(i) Find the value of p. [3]

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(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

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Q8

1 The cubic equation $7x^3 + 3x^2 + 5x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are α^{-1}, β^{-1}, γ^{-1}. [3]

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(b) Find the value of $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$. [2]

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(c) Find the value of $\alpha^{-3} + \beta^{-3} + \gamma^{-3}$. [2]

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Q9

7 The equation $x^3 + 2x^2 + x + 7 = 0$ has roots α, β, γ.

(i) Use the relation $x^2 = -7y$ to show that the equation

$$49y^3 + 14y^2 - 27y + 7 = 0$$

has roots $\frac{\alpha}{\beta\gamma}$, $\frac{\beta}{\gamma\alpha}$, $\frac{\gamma}{\alpha\beta}$. [4]

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(ii) Show that $\frac{\alpha^2}{\beta^2\gamma^2} + \frac{\beta^2}{\gamma^2\alpha^2} + \frac{\gamma^2}{\alpha^2\beta^2} = \frac{58}{49}$. [3]

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(iii) Find the exact value of $\frac{\alpha^3}{\beta^3\gamma^3} + \frac{\beta^3}{\gamma^3\alpha^3} + \frac{\gamma^3}{\alpha^3\beta^3}$. [2]

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Q10

6 The equation

$$x^3 - x + 1 = 0$$

has roots α, β, γ.

(i) Use the relation $x = y^{\frac{1}{3}}$ to show that the equation

$$y^3 + 3y^2 + 2y + 1 = 0$$

has roots α^3, β^3, γ^3. Hence write down the value of $\alpha^3 + \beta^3 + \gamma^3$. [3]

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Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(ii) Find the value of S_{-3}. [2]

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(iii) Show that $S_6 = 5$ and find the value of S_9. [4]

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Q11

9 A cubic equation $x^3 + bx^2 + cx + d = 0$ has real roots α, β and γ such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -\frac{5}{12},$$
$$\alpha\beta\gamma = -12,$$
$$\alpha^3 + \beta^3 + \gamma^3 = 90.$$

(i) Find the values of c and d. [3]

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(ii) Express $\alpha^2 + \beta^2 + \gamma^2$ in terms of b. [2]

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(iii) Show that $b^3 - 15b + 126 = 0$. [4]

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(iv) Given that $3 + i\sqrt{(12)}$ is a root of $y^3 - 15y + 126 = 0$, deduce the value of b. [2]

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Q12

4 It is given that the equation

$$x^3 - 21x^2 + kx - 216 = 0,$$

where k is a constant, has real roots a, ar and ar^{-1}.

(i) Find the numerical values of the roots. [6]

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(ii) Deduce the value of k. [2]

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Q13

6 The equation

$$9x^3 - 9x^2 + x - 2 = 0$$

has roots α, β, γ.

(i) Use the substitution $y = 3x - 1$ to show that $3\alpha - 1$, $3\beta - 1$, $3\gamma - 1$ are the roots of the equation

$$y^3 - 2y - 7 = 0.$$ [2]

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The sum $(3\alpha - 1)^n + (3\beta - 1)^n + (3\gamma - 1)^n$ is denoted by S_n.

(ii) Find the value of S_3. [2]

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(iii) Find the value of S_{-2}. [4]

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Q14

2 The roots of the equation

$$x^3 + px^2 + qx + r = 0$$

are α, 2α, 4α, where p, q, r and α are non-zero real constants.

(i) Show that

$$2p\alpha + q = 0. \qquad [4]$$

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(ii) Show that

$$p^3 r - q^3 = 0. \qquad [2]$$

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Q15

1 The roots of the cubic equation

$$x^3 - 5x^2 + 13x - 4 = 0$$

are α, β, γ.

(i) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3]

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(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

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Q16

7 By finding a cubic equation whose roots are α, β and γ, solve the set of simultaneous equations

$$\begin{aligned} \alpha + \beta + \gamma &= -1, \\ \alpha^2 + \beta^2 + \gamma^2 &= 29, \\ \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= -1. \end{aligned}$$

[8]

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Q17

1 The roots of the cubic equation $x^3 + 2x^2 - 3 = 0$ are α, β and γ.

(i) By using the substitution $y = \frac{1}{x^2}$, find the cubic equation with roots $\frac{1}{\alpha^2}$, $\frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$. [3]

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(ii) Hence find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$. [1]

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(iii) Find also the value of $\frac{1}{\alpha^2\beta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\gamma^2\alpha^2}$. [1]

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Q18

4 The cubic equation $2x^3 - 3x^2 + 4x - 10 = 0$ has roots α, β and γ.

(i) Find the value of $(\alpha + 1)(\beta + 1)(\gamma + 1)$. [4]

(ii) Find the value of $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$. [4]

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Q19

1 The roots of the cubic equation $2x^3 + x^2 - 7 = 0$ are α, β and γ. Using the substitution $y = 1 + \frac{1}{x}$, or otherwise, find the cubic equation whose roots are $1 + \frac{1}{\alpha}$, $1 + \frac{1}{\beta}$ and $1 + \frac{1}{\gamma}$, giving your answer in the form $ay^3 + by^2 + cy + d = 0$, where a, b, c and d are constants to be found. [4]

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Q20

8 The cubic equation

$$z^3 - z^2 - z - 5 = 0$$

has roots α, β and γ. Show that the value of $\alpha^3 + \beta^3 + \gamma^3$ is 19. [4]

Find the value of $\alpha^4 + \beta^4 + \gamma^4$. [2]

Show that the cubic equation with roots $\frac{\alpha - 1}{\alpha}$, $\frac{\beta - 1}{\beta}$ and $\frac{\gamma - 1}{\gamma}$ may be found using the substitution $z = \frac{1}{1 - x}$, and find this equation, giving your answer in the form $px^3 + qx^2 + rx + s = 0$, where p, q, r and s are constants to be determined. [4]

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[Turn over

2 Find the cubic equation with roots α, β and γ such that

$$\alpha + \beta + \gamma = 3,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 1,$$
$$\alpha^3 + \beta^3 + \gamma^3 = -30,$$

giving your answer in the form $x^3 + px^2 + qx + r = 0$, where p, q and r are integers to be found. [6]

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4 The roots of the cubic equation $x^3 - 7x^2 + 2x - 3 = 0$ are α, β and γ. Find the values of

(i) $\dfrac{1}{(\alpha\beta)(\beta\gamma)(\gamma\alpha)}$,

(ii) $\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\gamma\alpha}$,

(iii) $\dfrac{1}{\alpha^2\beta\gamma} + \dfrac{1}{\alpha\beta^2\gamma} + \dfrac{1}{\alpha\beta\gamma^2}$. [6]

Deduce a cubic equation, with integer coefficients, having roots $\dfrac{1}{\alpha\beta}$, $\dfrac{1}{\beta\gamma}$ and $\dfrac{1}{\gamma\alpha}$. [2]

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Q23

1 The quartic equation $x^4 - px^2 + qx - r = 0$, where p, q and r are real constants, has two pairs of equal roots. Show that $p^2 + 4r = 0$ and state the value of q. [6]

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5 The cubic equation $x^3 + px^2 + qx + r = 0$, where p, q and r are integers, has roots α, β and γ, such that

$$\alpha + \beta + \gamma = 15,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 83.$$

Write down the value of p and find the value of q. [3]

Given that α, β and γ are all real and that $\alpha\beta + \alpha\gamma = 36$, find α and hence find the value of r. [5]

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Q25

1 The equation $x^3 + px + q = 0$, where p and q are constants, with $q \neq 0$, has one root which is the reciprocal of another root. Prove that $p + q^2 = 1$. [5]

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Q26

11

EITHER

The roots of the quartic equation $x^4 + 4x^3 + 2x^2 - 4x + 1 = 0$ are α, β, γ and δ. Find the values of

(i) $\alpha + \beta + \gamma + \delta$, [1]

(ii) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$, [2]

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta}$, [2]

(iv) $\dfrac{\alpha}{\beta\gamma\delta} + \dfrac{\beta}{\alpha\gamma\delta} + \dfrac{\gamma}{\alpha\beta\delta} + \dfrac{\delta}{\alpha\beta\gamma}$. [2]

Using the substitution $y = x + 1$, find a quartic equation in y. Solve this quartic equation and hence find the roots of the equation $x^4 + 4x^3 + 2x^2 - 4x + 1 = 0$. [7]

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3 The cubic equation $x^3 - 2x^2 - 3x + 4 = 0$ has roots α, β, γ. Given that $c = \alpha + \beta + \gamma$, state the value of c. [1]

Use the substitution $y = c - x$ to find a cubic equation whose roots are $\alpha + \beta$, $\beta + \gamma$, $\gamma + \alpha$. [3]

Find a cubic equation whose roots are $\frac{1}{\alpha + \beta}$, $\frac{1}{\beta + \gamma}$, $\frac{1}{\gamma + \alpha}$. [2]

Hence evaluate $\frac{1}{(\alpha + \beta)^2} + \frac{1}{(\beta + \gamma)^2} + \frac{1}{(\gamma + \alpha)^2}$. [2]

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2 The roots of the equation $x^4 - 4x^2 + 3x - 2 = 0$ are α, β, γ and δ; the sum $\alpha^n + \beta^n + \gamma^n + \delta^n$ is denoted by S_n. By using the relation $y = x^2$, or otherwise, show that α^2, β^2, γ^2 and δ^2 are the roots of the equation

$$y^4 - 8y^3 + 12y^2 + 7y + 4 = 0. \quad [3]$$

State the value of S_2 and hence show that

$$S_8 = 8S_6 - 12S_4 - 72. \quad [3]$$

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2 The cubic equation $x^3 - px - q = 0$, where p and q are constants, has roots α, β, γ. Show that

(i) $\alpha^2 + \beta^2 + \gamma^2 = 2p$, [1]

(ii) $\alpha^3 + \beta^3 + \gamma^3 = 3q$, [2]

(iii) $6(\alpha^5 + \beta^5 + \gamma^5) = 5(\alpha^3 + \beta^3 + \gamma^3)(\alpha^2 + \beta^2 + \gamma^2)$. [3]

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5 The equation

$$8x^3 + 36x^2 + kx - 21 = 0,$$

where k is a constant, has roots $a - d$, a, $a + d$. Find the numerical values of the roots and determine the value of k. [8]

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Q31

1 The roots of the cubic equation $x^3 - 7x^2 + 2x - 3 = 0$ are α, β, γ. Find the values of

(i) $\alpha^2 + \beta^2 + \gamma^2$, [2]

(ii) $\alpha^3 + \beta^3 + \gamma^3$. [3]

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[Turn over

11 **EITHER**

The roots of the equation $x^4 - 3x^2 + 5x - 2 = 0$ are α, β, γ, δ, and $\alpha^n + \beta^n + \gamma^n + \delta^n$ is denoted by S_n. Show that

$$S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0. \quad [2]$$

Find the values of

(i) S_2 and S_4, [3]

(ii) S_3 and S_5. [6]

Hence find the value of

$$\alpha^2(\beta^3 + \gamma^3 + \delta^3) + \beta^2(\gamma^3 + \delta^3 + \alpha^3) + \gamma^2(\delta^3 + \alpha^3 + \beta^3) + \delta^2(\alpha^3 + \beta^3 + \gamma^3). \quad [3]$$

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7 A cubic equation has roots α, β and γ such that

$$\alpha + \beta + \gamma = 4,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 14,$$
$$\alpha^3 + \beta^3 + \gamma^3 = 34.$$

Find the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. [2]

Show that the cubic equation is

$$x^3 - 4x^2 + x + 6 = 0,$$

and solve this equation. [6]

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8 The cubic equation $x^3 - x^2 - 3x - 10 = 0$ has roots α, β, γ.

(i) Let $u = -\alpha + \beta + \gamma$. Show that $u + 2\alpha = 1$, and hence find a cubic equation having roots $-\alpha + \beta + \gamma$, $\alpha - \beta + \gamma$, $\alpha + \beta - \gamma$. [5]

(ii) State the value of $\alpha\beta\gamma$ and hence find a cubic equation having roots $\frac{1}{\beta\gamma}$, $\frac{1}{\gamma\alpha}$, $\frac{1}{\alpha\beta}$. [5]

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ROOTS OF POLYNOMIAL MS

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Q1

Question	Answer	Marks	Guidance
3(a)	$y = x^3$	**B1**	Correct substitution.
	$y^{\frac{4}{3}} - 2y - 1 = 0 \Rightarrow y^4 = (2y+1)^3 = 8y^3 + 12y^2 + 6y + 1$	**M1**	Obtains an equation not involving radicals.
	$y^4 - 8y^3 - 12y^2 - 6y - 1 = 0$	**A1**	
	$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = 8$	**B1 FT**	
		4	

Question	Answer	Marks	Guidance
3(b)	$\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3} = \frac{\alpha^3\beta^3\delta^3 + \alpha^3\beta^3\gamma^3 + \beta^3\gamma^3\delta^3 + \alpha^3\gamma^3\delta^3}{\alpha^3\beta^3\gamma^3\delta^3} = \frac{6}{-1}$	**M1** **A1 FT**	Relates $\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3}$ to coefficients.
	-6	**A1**	
		3	
3(c)	$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = 2(\alpha^3 + \beta^3 + \gamma^3 + \delta^3) + 4$	**M1**	Uses original equation.
	$= 20$	**A1**	
		2	

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Q2

2(a)	$S_1 = 2$	**B1**	
	$S_2 = S_1^{\,2} - 2(0)$	**M1**	Uses formula for sum of squares.
	$= 4$	**A1**	Correct answer implies M1A1.
		3	
2(b)(i)	$S_{n+3} = 2S_{n+2} - \frac{3}{2}S_n$	**B1**	CAO or as a single fraction.
		1	
2(b)(ii)	$S_4 = 2S_3 - \frac{3}{2}S_1 = 2(2S_2 - \frac{3}{2}S_0) - \frac{3}{2}S_1$	**M1**	Uses their recursive formula from part **(i)** to find S_4 $\left[S_3 = \frac{7}{2}\right]$.
	$= 4$	**A1**	
		2	
2(c)	$x = 2 - y$	**B1**	SOI
	$2(2-y)^3 - 4(2-y)^2 + 3 = 0$	**M1**	Makes *their* substitution.
	$2y^3 - 8y^2 + 8y - 3 = 0$	**A1**	OE but must be an equation.
		3	
2(d)	$\frac{\frac{8}{2}}{\frac{-3}{2}}$ OR Or use $2S_2 - 8S_1 + 8S_0 - 3S_{-1} = 0$ with substitution of their values	**M1**	Uses $\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} = \frac{\alpha'\beta' + \beta'\gamma' + \gamma'\alpha'}{\alpha'\beta'\gamma'}$.
	$= \frac{8}{3}$	**A1 FT**	FT from **2(c).**
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Q3

Question	Answer	Marks	Guidance
1	$b=-(\alpha+\beta+\gamma)=-3$	**B1**	
	$5=3^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$	**M1 A1**	Uses formula for sum of squares.
	$c=2$	**A1**	
	$6-3(5)+2(3)+3d=0$	**M1**	Uses original equation or formula for sum of cubes.
	[Equation is $x^3-3x^2+2x+1=0$] $d=1$	**A1**	
		6	

Q4

Question	Answer	Marks	Guidance
4(a)	$(-2)^2-2(3)$	**M1**	Uses formula for sum of squares.
	-2	**A1**	
		2	
4(b)	$\alpha^3+\beta^3+\gamma^3=-2(-2)-3(-2)-3(3)$	**M1**	Uses original equation or formula for sum of cubes.
	1	**A1**	AG
		2	

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4(c)	$(\alpha+r)^3=\alpha^3+3\alpha^2r+3\alpha r^2+r^3$	**B1**	Expands.
	$\sum_{r=1}^{n}\left((\alpha+r)^3+(\beta+r)^3+(\gamma+r)^3\right)=\sum_{r=1}^{n}\left(1+3(-2)r+3(-2)r^2+3r^3\right)$	**M1 A1**	Collects like terms and uses results from parts **(a)** and **(b)**.
	$n-6\left(\frac{1}{2}n(n+1)\right)-6\left(\frac{1}{6}n(n+1)(2n+1)\right)+\frac{3}{4}n^2(n+1)^2$ $n-3n(n+1)-n(n+1)(2n+1)+\frac{3}{4}n^2(n+1)^2$	**M1**	Applies formulae from MF19.
	$n+\frac{1}{4}n(n+1)\left(-12-4(2n+1)+3n(n+1)\right)$ $n+\frac{1}{4}n(n+1)\left(3n^2-5n-16\right)$	**M1 A1**	Simplifies.
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Q5

Question	Answer	Marks	Guidance
3(a)	$y=x^3\Rightarrow x=y^{\frac{1}{3}}$	**B1**	Substitutes.
	$y+cy^{\frac{1}{3}}+1=0\Rightarrow -c^3y=(y+1)^3=y^3+3y^2+3y+1$	**M1**	Correct attempt to eliminate cube root.
	$y^3+3y^2+\left(3+c^3\right)y+1=0$	**A1**	
		3	

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3(b)	$\alpha^3+\beta^3+\gamma^3=-3\ \ \alpha^3\beta^3+\beta^3\gamma^3+\gamma^3\alpha^3=3+c^3$	**B1 FT**	Using *their* answer to **(a)**.
	$\alpha^6+\beta^6+\gamma^6=(-3)^2-2\left(3+c^3\right)$	**M1**	$\alpha^6+\beta^6+\gamma^6=(\alpha^3+\beta^3+\gamma^3)^2-2(\alpha^3\beta^3+\beta^3\gamma^3+\gamma^3\alpha^3)$
	$=3-2c^3$	**A1**	AG
		3	
3(c)	$\alpha^3\beta^3\gamma^3=-1$	**B1**	If using *their* answer to **(a)** FT
	$\begin{vmatrix}1 & \alpha^3 & \beta^3\\ \alpha^3 & 1 & \gamma^3\\ \beta^3 & \gamma^3 & 1\end{vmatrix}=1-\left(\alpha^6+\beta^6+\gamma^6\right)+2\alpha^3\beta^3\gamma^3=2c^3-4$	**M1 A1**	Evaluates determinant.
	$2c^3-4=0$	**M1**	Sets determinant equal to zero.
	$c=\sqrt[3]{2}$	**A1**	
		5	

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Q6

Question	Answer	Marks	Guidance
1(a)	$d=1$	**B1**	
		1	
1(b)	$y=x+1\Rightarrow x=y-1$	**B1**	Uses correct substitution.
	$y^3+(b-3)y^2+(c-2b+3)y+b-c=0$	**M1 A1**	Substitutes and expands.
	Alternative method for question 1(b)		
	$(\alpha+1)(\beta+1)+(\alpha+1)(\gamma+1)+(\beta+1)(\gamma+1)=c-2b+3$	**B1**	
	$(\alpha+1+\beta+1+\gamma+1)=3-b,\ \ (\alpha+1)(\beta+1)(\gamma+1)=c-b$	**M1**	Using these relationships.
	$y^3+(b-3)y^2+(c-2b+3)y+b-c=0)$	**A1**	
		3	
1(c)	$\beta+1=-(b-3)$	**B1**	Uses sum of roots.
	$-(\alpha+1)(\alpha+1)=c-2b+3$	**B1**	Uses sum of products in pairs.
	$-(\alpha+1)(\beta+1)(\alpha+1)=-(b-c)$	**M1**	Applies product of roots.
	$\Rightarrow(c-2b+3)(b-3)=b-c$	**A1**	AG
		4	

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Q7

Question	Answer	Marks
2(a)	$y = x^2$	**B1**
	$6y^{\frac{3}{2}} + py - 3y^{\frac{1}{2}} - 5 = 0 \Rightarrow y^{\frac{1}{2}}(6y-3) = -py + 5$ $y(6y-3)^2 = (-py+5)^2 \Rightarrow y(36y^2 - 36y + 9) = p^2y^2 - 10py + 25$	**M1**
	$36y^3 - (p^2+36)y^2 + (10p+9)y - 25 = 0$	**A1**
		3
2(b)(i)	$\alpha^2 + \beta^2 + \gamma^2 = \dfrac{p^2+36}{36}$	**B1**
	$\dfrac{p^2+36}{36} = -\dfrac{2p}{6} \Rightarrow p^2 + 12p + 36 = 0$	**M1**
	$p = -6$	**A1**
		3
2(b)(ii)	$6(\alpha^3 + \beta^3 + \gamma^3) = 6(\alpha^2 + \beta^2 + \gamma^2) + 3(\alpha + \beta + \gamma) + 15$	**M1**
	$\alpha^3 + \beta^3 + \gamma^3 = 5$	**A1**
		2

Q8

Question	Answer	Marks
1(a)	$y = x^{-1}$	**B1**
	$7y^{-3} + 3y^{-2} + 5y^{-1} + 1 = 0 \Rightarrow y^3 + 5y^2 + 3y + 7 = 0$	**M1 A1**
		3
1(b)	$\alpha^{-2} + \beta^{-2} + \gamma^{-2} = (-5)^2 - 2(3) = 19$	**M1 A1**
1(c)	$\alpha^{-3} + \beta^{-3} + \gamma^{-3} = -5(19) - 3(-5) - 21 = -101$	**M1 A1**
		4

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Q9

Question	Answer	Marks	Guidance
7(i)	$\sqrt{-7y}(-7y)+2(-7y)+\sqrt{-7y}+7=0$ $\Rightarrow \sqrt{-7y}(-7y+1)=14y-7 \Rightarrow -7y(-7y+1)^2=(14y-7)^2$	**M1**	Uses given substitution and eliminates radical.
	$\Rightarrow 49y^3+14y^2-27y+7=0$	**A1**	AG
	$y=\frac{x^2}{-7}=\frac{x^2}{\alpha\beta\gamma}$	**M1**	Uses $\alpha\beta\gamma=-7$.
	So roots are $\frac{\alpha^2}{\alpha\beta\gamma}=\frac{\alpha}{\beta\gamma}, \frac{\beta^2}{\alpha\beta\gamma}=\frac{\beta}{\alpha\gamma}, \frac{\gamma^2}{\alpha\beta\gamma}=\frac{\gamma}{\alpha\beta}$	**A1**	AG
		4	
7(ii)	$\frac{\alpha}{\beta\gamma}+\frac{\beta}{\alpha\gamma}+\frac{\gamma}{\alpha\beta}=-\frac{2}{7}, \frac{1}{\gamma^2}+\frac{1}{\beta^2}+\frac{1}{\alpha^2}=-\frac{27}{49}$	**B1**	States sum of roots and $\alpha'\beta'+\alpha'\gamma'+\beta'\gamma'$.
	$\frac{\alpha^2}{\beta^2\gamma^2}+\frac{\beta^2}{\alpha^2\gamma^2}+\frac{\gamma^2}{\alpha^2\beta^2}=\left(-\frac{2}{7}\right)^2-2\left(-\frac{27}{49}\right)=\frac{58}{49}$	**M1 A1**	Uses $\alpha'^2+\beta'^2+\gamma'^2=(\alpha'+\beta'+\gamma')^2-2(\alpha'\beta'+\alpha'\gamma'+\beta'\gamma')$ AG
		3	
7(iii)	$49\left(\frac{\alpha^3}{\beta^3\gamma^3}+\frac{\beta^3}{\alpha^3\gamma^3}+\frac{\gamma^3}{\alpha^3\beta^3}\right)=-14\left(\frac{58}{49}\right)+27\left(-\frac{2}{7}\right)-21$	**M1**	Uses $49\alpha'^3=-14\alpha'^2+27\alpha'-7$.
	$\Rightarrow \frac{\alpha^3}{\beta^3\gamma^3}+\frac{\beta^3}{\alpha^3\gamma^3}+\frac{\gamma^3}{\alpha^3\beta^3}=-\frac{317}{343}$	**A1**	
		2	

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Q10

Question	Answer	Marks	Guidance
6(i)	$y=(y+1)^3$	**M1**	Obtains an equation in y not involving radicals.
	$y=y^3+3y^2+3y+1\Rightarrow y^3+3y^2+2y+1=0$	**A1**	AG
	$S_3=-3$	**B1**	
		3	
6(ii)	$S_{-3}=\dfrac{\alpha^3\beta^3+\beta^3\gamma^3+\alpha^3\gamma^3}{\alpha^3\beta^3\gamma^3}=\dfrac{2}{-1}=-2$	**M1 A1**	
		2	
6(iii)	$S_6=(-3)^2-2(2)=5$	**M1 A1**	Uses $(\Sigma\alpha)^2=\Sigma\alpha^2+2\sum_{\alpha\neq\beta}\alpha\beta$. AG.
	$S_9=-3S_6-2S_3-3=-3(5)-2(-3)-3=-12$	**M1 A1**	Sums $y^3-3y^2+2y-1=0$ for $y=\alpha^3,\beta^3,\gamma^3$.
		4	

Q11

Question	Answer	Marks	Guidance
9(i)	$\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}=\dfrac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma}\Rightarrow c=\alpha\beta+\beta\gamma+\alpha\gamma=5$	**M1 A1**	Uses $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}=\dfrac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma}$.
	$d=12$	**B1**	
		3	

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9(ii)	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$	**M1**	
	$(\alpha+\beta+\gamma)^2-10=b^2-10$	**A1**	
	Alternative method for 9(ii)		
	$90+bS_2-5b+36=0\Rightarrow S_2=\dfrac{5b-126}{b}$	**M1 A1**	Sums over roots
		2	

9(iii)	$x^3+bx^2+5x+12=0$ $\qquad(-b=\alpha+\beta+\gamma)$	**M1**	Formulates equation.
	$90+b\left(b^2-10\right)-5b+36=0$	**M1 A1**	Sums over roots and uses **9(ii)**
	$b^3-15b+126=0$	**A1**	
	Alternative method 1 for 9(iii)		
	$(\alpha+\beta+\gamma)^3=\alpha^3+\beta^3+\gamma^3+3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)-3\alpha\beta\gamma$	**M1 A1**	Uses expansion of $(\alpha+\beta+\gamma)^3$
	$-b^3=90-15b+36$	**(M1)**	Substitutes known values.
	$b^3-15b+126=0$	**A1**	
	Alternative method 2 for 9(iii)		
	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)=b^2-10$	**M1 A1**	Finds sum of squares in terms of b.
	$\dfrac{5b-126}{b}=b^2-10\Rightarrow b^3-15b+126=0$	**M1 A1**	Equates expressions.
		4	
9(iv)	Real root is $b=-6$	**M1 A1**	Gives the real root for b.

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Q12

Question	Answer	Marks	Guidance
4(i)	$\alpha\beta\gamma = a^3 = 216 \Rightarrow a = 6$	**M1 A1**	Uses product of roots
	$a + ar + ar^{-1} = 21$ $6\left(1 + r + r^{-1}\right) = 21$	**M1**	Uses sum of roots
	$2r^2 - 5r + 2 = 0 \Rightarrow r = 2$ or $r = 0.5$	**M1 A1**	Substitutes for a and solves quadratic
	Roots are 6, 12, 3	**A1**	
		6	
4(ii)	$k = \alpha\beta + \alpha\gamma + \beta\gamma = 6(12) + 6(3) + 12(3) = 126$	**M1 A1**	Or finds coefficient of x in $(x-3)(x-6)(x-12)$. Or substitutes root into equation
		2	

Q13

Question	Answer	Marks	Guidance
6(i)	Substitutes $x = \dfrac{y+1}{3}$	**M1**	Accept substitution of $y = 3x - 1$ into given equation and derivation of equation in x.
	Obtains the given result	**A1**	AG.
6(ii)	$S_3 = 2S_1 + 7 \times 3$	**M1**	Uses $y^3 = 2y + 7$. Or uses formula for $\Sigma\,(3\alpha - 1)^3$
	$= 21$	**A1**	

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6(iii)	$S_{-1}=\dfrac{(3\alpha-1)(3\beta-1)+(3\alpha-1)(3\gamma-1)+(3\beta-1)(3\gamma-1)}{(3\alpha-1)(3\beta-1)(3\gamma-1)}=\dfrac{-2}{7}$.	**M1 A1**	Award M1A1 if $S_{-1}=-\dfrac{2}{7}$ written down directly.
	$7S_{-2}=S_1-2S_{-1}$	**M1**	Uses $7y^{-2}=y-2y^{-1}$.
	$s_{-2}=\dfrac{4}{49}$	**A1**	
	$\Sigma\dfrac{1}{\quad}=\dfrac{\Sigma(3\alpha-1)^2(3\beta-1)^2}{\quad}$	**M1 A1**	**Alt method**: Finds cubic with roots $\dfrac{1}{\quad}$, etc. M1 $7z^3+2z^2-1=0$ A1 Uses $S_2=(S_1)^2-2x\Sigma\alpha\beta$ M1 $=\dfrac{4}{49}$ A1
	$\dfrac{(\Sigma(3\alpha-1)(3\beta-1))^2-2(3\alpha-1)(3\beta-1)(3\gamma-1)(\Sigma(3\alpha-1))}{(3\alpha-1)^2(3\beta-1)^2(3\gamma-1)^2}$	**M1**	
	$=\dfrac{(-2)2-2(7)(0)}{7^2}=\dfrac{4}{49}$	**A1**	
		8	

Q14

Question	Answer	Marks	Guidance
2(i)	$\alpha+2\alpha+4\alpha=-p$	**B1**	Sum of roots.
	$2\alpha^2+4\alpha^2+8\alpha^2=q$	**B1**	Sum of products in pairs.
	$\dfrac{14\alpha^2}{7\alpha}=-\dfrac{q}{p}$	**M1**	Combines equations.
	$\Rightarrow 2p\alpha+q=0$	**A1**	Verifies result (AG).
		4	

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2(ii)	$8\alpha^3 = -r$	**B1**	Product of roots.
	$\Rightarrow r = \dfrac{q^3}{p^3} \Rightarrow p^3 r - q^3 = 0$	**B1**	Verifies result (AG).
		2	

Q15

Question	Answer	Marks	Guidance
1(i)	$\alpha + \beta + \gamma = 5$, $\alpha\beta + \alpha\gamma + \beta\gamma = 13$	**B1**	Sum of roots and $\alpha\beta + \alpha\gamma + \beta\gamma$. SOI
	$\alpha^2 + \beta^2 + \gamma^2 = 5^2 - 2(13)$	**M1**	Uses $\sum\alpha^2 = (\sum\alpha)^2 - 2(\sum\alpha\beta)$
	$= -1$	**A1**	www
		3	
1(ii)	$\alpha^3 + \beta^3 + \gamma^3 = 5(\alpha^2 + \beta^2 + \gamma^2) - 13(\alpha + \beta + \gamma) + 12$	**M1**	Uses $\alpha^3 = 5\alpha^2 - 13\alpha + 4$.
	$= 5(-1) - 13(5) + 12 = -58$	**A1**	
	Alt method: Use formula e.g. $\sum\alpha^3 = (\sum\alpha)(\sum\alpha^2 - \sum\alpha\beta) + 3\alpha\beta\gamma$ Or $(\sum\alpha)^3 - 3(\sum\alpha)(\sum\alpha\beta) + 3\alpha\beta\gamma$		
		2	

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Q16

Question	Answer	Marks	Guidance
7	$2\sum\alpha\beta = 1-29 \Rightarrow \sum\alpha\beta = -14$	**M1A1**	
	$\dfrac{\sum\alpha\beta}{\alpha\beta\gamma} = \dfrac{-14}{\alpha\beta\gamma} = -1 \Rightarrow \alpha\beta\gamma = 14$	**M1A1 FT**	
	$\Rightarrow x^3 + x^2 - 14x - 14 = 0$	**A1**	
	$\Rightarrow (x+1)\left(x^2 - 14\right)$	**M1A1**	Attempt to factorise cubic
	⇒Solution is –1, in $\pm\sqrt{14}$ any order. Accept ±3.74 (awrt) SR **B1** for correct roots without working	**A1**	
	Total:	**8**	

Q17

Question	Answer	Marks	Guidance
1(i)	$y = \dfrac{1}{x^2} \Rightarrow x = \dfrac{1}{\sqrt{y}}$	**M1**	Rearranging to make x the subject
	$\dfrac{1}{y\sqrt{y}} + \dfrac{2}{y} - 3 = 0 \Rightarrow \dfrac{1}{y\sqrt{y}} = 3 - \dfrac{2}{y}$	**M1**	Substituting and squaring
	$\Rightarrow 9y^3 - 12y^2 + 4y - 1 = 0$ SR **B1** for finding cubic by manipulating roots	**A1**	OE
	Total:	**3**	
1(ii)	$\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2} = \dfrac{12}{9}$ or $\dfrac{4}{3}$	**B1FT**	
	Total:	**1**	
1(iii)	$\dfrac{1}{\alpha^2\beta^2} + \dfrac{1}{\beta^2\gamma^2} + \dfrac{1}{\gamma^2\alpha^2} = \dfrac{4}{9}$	**B1FT**	
	Total:	**1**	

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Q18

Question	Answer	Marks	Guidance
4(i)	$\alpha+\beta+\gamma=\frac{3}{2}\quad \alpha\beta+\beta\gamma+\gamma\alpha=2\quad \alpha\beta\gamma=5+\beta+\gamma=$ $\frac{3}{2}\alpha\beta+\beta\gamma+\gamma\alpha=2\alpha\beta\gamma=5$	**B1**	(Can be awarded in (ii) if not seen here) SOI
	$(\alpha+1)(\beta+1)(\gamma+1)=\alpha\beta\gamma+(\alpha\beta+\beta\gamma+\gamma\alpha)+$ $(\alpha+\beta+\gamma)+1$	**M1A1**	Multiply out **and** group for **M1**
	$=5+2+1\frac{1}{2}+1=9\frac{1}{2}$	**A1FT**	Alt method: Let $x=y-1$ **M1** Sub and expand $2y^3-9y^2\ 16y-19=0$ **M1**, **A1** Product of roots = 19/2 **A1**
		4	
4(ii)	$(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)=\left(1\frac{1}{2}-\alpha\right)\left(1\frac{1}{2}-\beta\right)\left(1\frac{1}{2}-\gamma\right)$	**M1**	Alt methods: $=(\sum\alpha)(\sum\alpha\beta)-\alpha\beta\gamma$ or $\sum\alpha^2\sum\alpha+2\alpha\beta\gamma-\sum\alpha^3$
	$=\frac{27}{8}-\frac{9}{4}(\alpha+\beta+\gamma)+\frac{3}{2}(\alpha\beta+\beta\gamma+\gamma\alpha)-\alpha\beta\gamma$	**A1**	
	$=\frac{27}{8}-\frac{9}{4}\times\frac{3}{2}+\frac{3}{2}\times2-5=-2$	**M1A1**	
		4	

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Q19

1	$y = 1 + \frac{1}{x} \Rightarrow x = \frac{1}{y-1}$	**M1**
	$\frac{2}{(y-1)^3} + \frac{1}{(y-1)^2} - 7 = 0 \Rightarrow 2 + (y-1) - 7(y-1)^3 = 0$	**A1**
	$\Rightarrow 7(y^3 - 3y^2 + 3y - 1) - y + 1 - 2 = 0 \Rightarrow 7y^3 - 21y^2 + 20y - 8 = 0$	**M1A1** [4]
	ALT METHOD: ∑α, ∑αβ, αβγM1 A1, ∑(1+1/α) etc M1 A1√	

Q20

8	Let S_n denote $\alpha^n + \beta^n + \gamma^n$.	
	$S_2 = S_1^2 - 2\sum\alpha\beta = 1^2 - 2\times(-1) = 3$	**M1A1**
	$S_3 = S_2 + S_1 + 15 = 3 + 1 + 15 = 19$ (**AG**)	**M1A1** [4]
	Alt method: $\sum\alpha = 1$, Σαβ= -1, and αβγ = 5 **M1, A1**	
	$S_3 = S_1^3 - 3\Sigma\alpha\,\Sigma\alpha\beta + 3\alpha\beta\gamma = 19$ **M1, A1**	
	$S_4 = S_3 + S_2 + 5S_1 = 19 + 3 + 5 = 27$	**M1A1** [2]
	$x = \frac{z-1}{z} = 1 - \frac{1}{z} \Rightarrow \frac{1}{z} = 1 - x \Rightarrow z = \frac{1}{1-x}$	**B1**
	Hence $\frac{1}{(1-x)^3} - \frac{1}{(1-x)^2} - \frac{1}{(1-x)} - 5 = 0$ is the required equation.	**M1**
	$\Rightarrow 1 - (1-x) - (1-x)^2 - 5(1-x)^3 = 0 \Rightarrow \ldots \Rightarrow 5x^3 - 16x^2 + 18x - 6 = 0$	**M1A1** [4]

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Q21

2	$2\sum\alpha\beta = 9-1 \Rightarrow \sum\alpha\beta = 4$	**M1A1**	
	Use of, e.g.: $\sum\alpha^3 - 3\alpha\beta\gamma = \sum\alpha\left(\sum\alpha^2 - \sum\alpha\beta\right)$	**M1**	
	or $\left(\sum\alpha\right)^3 = \sum\alpha^3 + 3\sum\alpha\sum\alpha\beta - 3\alpha\beta\gamma$		
	Correct substitution in formula	**A1**	
	$\Rightarrow \alpha\beta\gamma = -7$	**A1**	
	Required cubic equation is $x^3 - 3x^2 + 4x + 7 = 0$ *must see final equation*	**A1**√	[6]
	ALT METHOD: $S_3 - 3S_2 + 4S_1 + 3r = 0$ **M1** $3r = 30 + 3 \times 1 - 4 \times 3$ **A1** $r=7$ **A1**		

Q22

4		$\alpha+\beta+\gamma = 7$, $\alpha\beta+\beta\gamma+\gamma\alpha = 2$, $\alpha\beta\gamma = 3$ (Stated or implied by working.)	B1
	(i)	$\dfrac{1}{(\alpha\beta)(\beta\gamma)(\gamma\alpha)} = \dfrac{1}{(\alpha\beta\gamma)^2} = \dfrac{1}{9}$	B1
	(ii)	$\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\gamma\alpha} = \dfrac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \dfrac{7}{3}$	M1A1
	(iii)	$\dfrac{1}{\alpha^2\beta\gamma} + \dfrac{1}{\alpha\beta^2\gamma} + \dfrac{1}{\alpha\beta\gamma^2} = \dfrac{1}{\alpha\beta\gamma}\left(\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}\right) = \dfrac{1}{\alpha\beta\gamma}\left(\dfrac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}\right) = \dfrac{2}{9}$	M1A1 (6)
		$\Rightarrow x^3 - \dfrac{7}{3}x^2 + \dfrac{2}{9}x - \dfrac{1}{9} = 0 \Rightarrow 9x^3 - 21x^2 + 2x - 1 = 0$	M1A1√ (2) **Total: 8**

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Q23

1	i $2\alpha + 2\beta = 0$ ii $\alpha^2 + 4\alpha\beta + \beta^2 = -p$ iii $2\alpha^2\beta + 2\alpha\beta^2 = -q$ iv $\alpha^2\beta^2 = -r$ All four correct (Any two correct) Use of $\beta = -\alpha$ in ii, iii or iv $\Rightarrow p = 2\alpha^2$ and $\alpha^4 = -r \Rightarrow p^2 + 4r = 0$ (AG) and $q = 0$ (CAO)	B2 (B1) M1 M1A1 B1 (6) **Total 6**

Q24

5	$\alpha + \beta + \gamma = -p = 15 \Rightarrow p = -15$	**B1**
	$2(\alpha\beta + \beta\gamma + \gamma\alpha) = (\alpha + \beta + \gamma)^2 - (\alpha^2 + \beta^2 + \gamma^2) = 2q$	**M1**
	$\Rightarrow q = \frac{1}{2}(225 - 83) = 71$	**A1** [3]
	$\frac{36}{\alpha} = 15 - \alpha \quad (= [\beta + \gamma])$	**M1**
	$\Rightarrow \alpha^2 - 15\alpha + 36 = 0 \Rightarrow \alpha = 3$, $\alpha \neq 12$, ***e.g.*** since $12^2 > 83$ or other reason	**M1A1**
	$\beta\gamma = 71 - 36 = 35$	**B1**
	$\Rightarrow r = -\alpha\beta\gamma = -3 \times 35 = -105$ (extra answer penalised)	**A1** [5] **Total 8**

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Q25

1	Let roots be α, α^{-1} and β $\Rightarrow \alpha + \alpha^{-1} + \beta = 0$ (1) Product of roots $\Rightarrow \beta = -q$ (2) Sum of products in pairs $\Rightarrow 1 + \beta(\alpha + \alpha^{-1}) = p$ (3) From (1) and (3) $1 - \beta^2 = p$ Using (2) $1 - q^2 = p$ or $p + q^2 = 1$	Any of three for M1 A1 for another. A1 for a third Wrong sign in (2) scores M1A0A1M1A0	M1 A1 A1 M1 A1 [5]

Q26

11E (i)	$\alpha+\beta+\gamma+\delta=-4$	B1 (1)
(ii)	$\alpha^2+\beta^2+\gamma^2+\delta^2=(-4)^2-2\times 2=12$	M1A1 (2)
(iii)	$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{-(-4)}{1}=4$	M1A1 (2)
(iv)	$\frac{\alpha}{\beta\gamma\delta}+\frac{\beta}{\alpha\gamma\delta}+\frac{\gamma}{\alpha\beta\delta}+\frac{\delta}{\alpha\beta\gamma}=\frac{\alpha^2+\beta^2+\gamma^2+\delta^2}{\alpha\beta\gamma\delta}=\frac{12}{1}=12$	M1A1 (2)
	$y=x+1\Rightarrow x=y-1$	
	$(y-1)^4+4(y-1)^3=y^4-6y^2+8y-3$	M1A1
	$2(y-1)^2-4(y-1)+1=2y^2-8y+7$	A1
	$\Rightarrow x^4+4x^3+2x^2-4x+1=y^4-4y^2+4=0$	A1
	$(y^2-2)^2=0\Rightarrow y=\pm\sqrt{2}$ (twice).	A1
	$\Rightarrow x=\pm\sqrt{2}-1$ **(twice).** (Some indication of four roots for fina	M1A1 (7) **[14]**

Q27

3	Uses $\sum\alpha=\frac{-b}{a}$.	$c=2$	B1	1	
	Uses substitution	$(\alpha+\beta=c-\gamma$ etc.$)\Rightarrow y=c-x\Rightarrow x=c-y$	M1		
		$(2-y)^3-2(2-y)^2-3(2-y)+4=0$ …(their c)	M1		
	to obtain required cubic equation.	$\Rightarrow y^3-4y^2+y+2=0$	A1		
	Obtains equation whose roots are reciprocals of those in previous cubic equation.	Uses $z=y^{-1}$ to obtain $2z^3+z^2-4z+1=0$	M1A1		
	Uses $\sum\alpha^2=(\sum\alpha)^2-2\sum\alpha\beta$	$\sum\frac{1}{(\alpha+\beta)^2}=\left(\frac{1}{2}\right)^2-2(-2)=4\frac{1}{4}$	M1A1	2	**[8]**

Q28

2	Makes substitution.	$y^2-4y+3y^{\frac{1}{2}}-2=0$	M1		
	Squares.	$\Rightarrow 9y=4+y^4+16y^2-4y^2+16y-8y^3$	M1		
		(N.B. Must see both terms in y^2.)			
	Obtains result.	$\Rightarrow y^4-8y^3+12y^2+7y+4=0$ (AG)	A1	3	
		$S_2=0^2-2\times(-4)=8$	B1		
		$S_8=8S_6-12S_4-7S_2-16$	M1		
		$\Rightarrow S_8=8S_6-12S_4-56-16=8S_6-12S_4-72$ (AG)	A1	3	**[6]**
		Alternatively – for final two marks.			
		$S_2=8$, $S_3=-9$, $S_4=40$, $S_5=-60$, S_6=203 , $S_7=-378$ $S_8=1072$ (generated by substitution of roots in equations and summing.) Then $8S_6-12S_4-72=1624-480-72=1072=S_8$ M1 requires a complete method, A1 if all correct.			

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Q29

2 (i)	Finds S_2.	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ $=0-2(-p)=2p$ (AG)	B1	1	
(ii)	Finds S_3.	$\alpha^3+\beta^3+\gamma^3=p\sum\alpha+3q=0+3q=3q$ (AG)	M1A1	2	
(iii)	Finds S_5.	$\alpha^5+\beta^5+\gamma^5=p\sum\alpha^3+q\sum\alpha^2$ $=p.3q+q.2p=5pq$ $\Rightarrow 6\sum\alpha^5=30\text{pq}=5\sum\alpha^3\sum\alpha^2$ (AG)	M1 A1 A1	3	**[6]**

Q30

5	Uses sum of roots	$\sum\alpha=3a=-\frac{36}{8}=-\frac{9}{2}\Rightarrow a=-\frac{3}{2}$	M1A1		
	Uses product of roots	$\alpha\beta\gamma=a(a^2-d^2)=\frac{21}{8}$	M1		
	Substitutes for a,	$\frac{9}{4}-d^2=-\frac{2}{3}\times\frac{21}{8}=-\frac{7}{4}$	M1		
	and solves	$\Rightarrow d^2=4\Rightarrow d=\pm2$	A1		
		Roots are $-\frac{7}{2},-\frac{3}{2},\frac{1}{2}$	A1		
	Uses sum of products in pairs. (Or expands $(2x+7)(2x+3)(2x+1)$ and equates coefficients)	$\sum\alpha\beta=\frac{21}{4}-\frac{7}{4}-\frac{3}{4}=\frac{k}{8}$ $\Rightarrow k=22$	M1 A1	**(8)**	**[8]**

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Q31

1	States $\sum\alpha$ and $\sum\alpha\beta$ Uses formula for correctly.	$\sum\alpha=7\quad\sum\alpha\beta=2$ $\sum\alpha^2=7^2-2\times2=45$	B1 B1	2	
	Uses formula for $\sum\alpha^3$ to obtain result.	$\sum\alpha^3=7\sum\alpha^2-2\sum\alpha+9$ $=315\text{-}14\text{+}9=310$	M1 A1A1	3	**[5]**

Q32

11	**EITHER** Substitute α into equation. Multiply by α^n. Obtain result.	α is a root $\Rightarrow \alpha^4 - 3\alpha^2 + 5\alpha - 2 = 0$ $\Rightarrow \alpha^{n+4} - 3\alpha^{n+2} + 5\alpha^{n+1} - 2\alpha^n = 0$ Repeat for β, γ, δ and sum $\Rightarrow S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0$ (AG)	M1 A1	2	
(i)	Uses $\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$ Finds S_4 from formula.	$S_2 = 0 - 2 \times (-3) = 6$ $S_4 = 3 \times 6 - 5 \times 0 + 2 \times 4 = 26$	B1 M1A1	3	
(ii)	$S_{-1} = \dfrac{\sum \alpha\beta\gamma}{\alpha\beta\gamma\delta}$ Finds S_3 from formula. Finds S_5 from formula.	$S_{-1} = \dfrac{-5}{-2} = \dfrac{5}{2}$ $S_3 = 3 \times 0 - 5 \times 4 + 2 \times \dfrac{5}{2} = -15$ $S_5 = 3 \times (-15) - 5 \times 6 + 2 \times 0 = -75$ $\sum \alpha^2\beta^3 = S_2S_3 - S_5$ $= 6 \times (-15) - (-75) = -15$	M1A1 M1A1 M1A1 M1 M1A1	6 3	**[14]**

Q33

7		$(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) = \alpha^2+\beta^2+\gamma^2 \Rightarrow \sum \alpha\beta = 1$ **Either** Required equation is $x^3 - 4x^2 + x + c = 0$ $\Rightarrow \sum \alpha^3 - 4\sum a^2 + 4 + 3c = 0$ $\Rightarrow 3c = 56 - 34 - 4 = 18 \Rightarrow c = 6$ (AG) **Or** $\alpha^3+\beta^3+\gamma^3 - 3\alpha\beta\gamma = (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha)$ (or some other appropriate identity, e.g. $(\alpha+\beta+\gamma)^3 = \alpha^3+\beta^3+\gamma^3+3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)-3\alpha\beta\gamma)$ $\Rightarrow \ldots \Rightarrow \alpha\beta\gamma = -6$ $\Rightarrow x^3 - 4x^2 + x + 6 = 0$ (AG) $\Rightarrow (x+1)(x-2)(x-3) = 0 \Rightarrow x = -1, 2, 3.$	M1A1 M1 M1 A1 (M1) (M1A1) A1 M1A1	2 6	**[8]**

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Q34

8 (i)	Deduces initial result.	$u = -\alpha + \beta + \gamma \Rightarrow u + 2\alpha = \alpha + \beta + \gamma = 1$	M1A1		
	Substitutes into cubic equation.	$\Rightarrow \alpha = \left(\frac{1-u}{2}\right)$	M1		
	Deduces new cubic equation.	$\Rightarrow \left(\frac{1-u}{2}\right)^3 - \left(\frac{1-u}{2}\right)^2 - 3\left(\frac{1-u}{2}\right) - 10 = 0$	A1		
		$\Rightarrow \ldots \Rightarrow u^3 - u^2 - 13u + 93 = 0$	A1	5	
(ii)	Deduces initial result.	$\alpha\beta\gamma = 10$	B1		
		$\Rightarrow v = \frac{1}{\beta\gamma} \Rightarrow \frac{v}{\alpha} = \frac{1}{\alpha\beta\gamma} = \frac{1}{10} \Rightarrow \alpha = 10v$	M1A1		
	Substitutes into cubic equation.	$(10v)^3 - (10v)^2 - 3(10v) - 10 = 0$	M1		
	Deduces new cubic equation.	$\Rightarrow 100v^3 - 10v^2 - 3v - 1 = 0$	A1	5	**[10]**

Alternatively:

8 (i) **For final 3 marks in (i):**

Award M1 for an attempt at formulae for all three coefficients.
A1 for any two correct.
A1 for completion

Let equation be $u^3 + bu^2 + cu + d = 0$.

$-b = \sum\alpha = 1 \Rightarrow b = -1$

$c = 4\sum\alpha\beta - (\sum\alpha)^2$

$= 4 \times (-3) - 1^2 = -13$

$-d = 4\sum\alpha\sum\alpha\beta - \left(\sum\alpha\right)^3 - 8\alpha\beta\gamma$

$= 4 \times 1 \times (-3) - 1^3 - 8 \times 10 = -93$

So $u^3 - u^2 - 13u + 93 = 0$ (5)

(ii) **For final 4 marks in (ii):**

Award M1 for an attempt at formulae for all three coefficients.
A1 for any one correct.
A1 for a second one correct.
A1 for completion.

Let equation be $v^3 + bv^2 + cv + d = 0$.

$-b = \frac{\Sigma\alpha}{\alpha\beta\gamma} = \frac{1}{10} \Rightarrow b = -\frac{1}{10}$

$c = \frac{\Sigma\alpha\beta}{(\alpha\beta\gamma)^2} = \frac{-3}{10^2} = -\frac{3}{100}$

$-d = \frac{1}{(\alpha\beta\gamma)^2} = \frac{1}{10^2} \Rightarrow d = -\frac{1}{100}$

So $v^3 - \frac{1}{10}v^2 - \frac{3}{100}v - \frac{1}{100} = 0$

or $100v^3 - 10v^2 - 3v - 1 = 0$. (5) **[10]**

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RATIONAL FUNCTIONS

Craftear

Q1

7 The curve C has equation $y = \dfrac{x^2 + x + 9}{x + 1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [4]

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(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\frac{x^2+x+9}{x+1}\right|$ and find the set of values of x for which $2|x^2+x+9| > 13|x+1|$. [5]

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Q22

7 The curve C has equation $y = \dfrac{x^2 - x - 3}{1 + x - x^2}$.

(a) Find the equations of the asymptotes of C. [2]

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(b) Find the coordinates of any stationary points on C. [3]

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(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\frac{x^2 - x - 3}{1 + x - x^2}\right|$ and find in exact form the set of values of x for which $\left|\frac{x^2 - x - 3}{1 + x - x^2}\right| < 3$. [6]

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Q3

7 The curve C has equation $y = \dfrac{4x+5}{4-4x^2}$.

(a) Find the equations of the asymptotes of C. [2]

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(b) Find the coordinates of any stationary points on C. [4]

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(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\frac{4x+5}{4-4x^2}\right|$ and find in exact form the set of values of x for which $4|4x+5| > 5|4-4x^2|$. [6]

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Q4

6 The curve C has equation $y = \frac{x^2}{x-3}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that there is no point on C for which $0 < y < 12$. [4]

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(c) Sketch C. [2]

(d) **(i)** Sketch the graphs of $y = \left|\frac{x^2}{x-3}\right|$ and $y = |x| - 3$ on a single diagram, stating the coordinates of the intersections with the axes. [4]

(ii) Use your sketch to find the set of values of c for which $\left|\frac{x^2}{x-3}\right| \leqslant |x| + c$ has no solution. [1]

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Q55

6 The curve C has equation $y = \dfrac{x^2 + x - 1}{x - 1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that there is no point on C for which $1 < y < 5$. [4]

Craftear

(c) Find the coordinates of the intersections of C with the axes, and sketch C. [3]

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(d) Sketch the curve with equation $y = \left| \frac{x^2 + x - 1}{x - 1} \right|$. [2]

Q66

6 Let a be a positive constant.

(a) The curve C_1 has equation $y = \dfrac{x-a}{x-2a}$. [2]

Sketch C_1.

The curve C_2 has equation $y = \left(\dfrac{x-a}{x-2a}\right)^2$. The curve C_3 has equation $y = \left|\dfrac{x-a}{x-2a}\right|$.

(b) (i) Find the coordinates of any stationary points of C_2. [3]

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(ii) Find also the coordinates of any points of intersection of C_2 and C_3. [3]

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(c) Sketch C_2 and C_3 on a single diagram, clearly identifying each curve. Hence find the set of values of x for which $\left(\frac{x-a}{x-2a}\right)^2 \leqslant \left|\frac{x-a}{x-2a}\right|$. [5]

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Q7

1 Let a be a positive constant.

(a) Sketch the curve with equation $y = \frac{ax}{x+7}$. [2]

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(b) Sketch the curve with equation $y = \left|\frac{ax}{x+7}\right|$ and find the set of values of x for which $\left|\frac{ax}{x+7}\right| > \frac{a}{2}$.

[4]

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Q88

3 The curve C has equation $y = \dfrac{x^2}{2x+1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [3]

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(c) Sketch C. [3]

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Q9

6 The curve C has equation $y = \dfrac{10 + x - 2x^2}{2x - 3}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that C has no turning points. [3]

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(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\frac{10+x-2x^2}{2x-3}\right|$ and find in exact form the set of values of x for which $\left|\frac{10+x-2x^2}{2x-3}\right| < 4$. [6]

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Q10

4 The line $y = 2x + 1$ is an asymptote of the curve C with equation

$$y = \frac{x^2 + 1}{ax + b}.$$

(i) Find the values of the constants a and b. [3]

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(ii) State the equation of the other asymptote of C. [1]

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(iii) Sketch C. [Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points.] [3]

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Q11

10 The curves C_1 and C_2 have equations

$$y = \frac{ax}{x+5} \quad \text{and} \quad y = \frac{x^2 + (a+10)x + 5a + 26}{x+5}$$

respectively, where a is a constant and $a > 2$.

(i) Find the equations of the asymptotes of C_1. [2]

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(ii) Find the equation of the oblique asymptote of C_2. [2]

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(iii) Show that C_1 and C_2 do not intersect. [2]

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(iv) Find the coordinates of the stationary points of C_2. [3]

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(v) Sketch C_1 and C_2 on a single diagram. [You do not need to calculate the coordinates of any points where C_2 crosses the axes.] [3]

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Q12

6 The curve C has equation

$$y = \frac{x^2}{kx - 1},$$

where k is a positive constant.

(i) Obtain the equations of the asymptotes of C. [3]

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(ii) Find the coordinates of the stationary points of C. [3]

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(iii) Sketch *C*. [3]

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Q13

6 The curve C has equation

$$y = \frac{x^2 + b}{x + b},$$

where b is a positive constant.

(i) Find the equations of the asymptotes of C. [3]

(ii) Show that C does not intersect the x-axis. [1]

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(iii) Justifying your answer, find the number of stationary points on C. [2]

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(iv) Sketch C. Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points. [3]

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Q14

4 The curve C has equation

$$y = \frac{x^2 + 7x + 6}{x - 2}.$$

(i) Find the coordinates of the points of intersection of C with the axes. [2]

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(ii) Find the equation of each of the asymptotes of C. [3]

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(iii) Sketch C. [3]

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Q15

6 The curve C has equation

$$y = \frac{x^2 + ax - 1}{x + 1},$$

where a is constant and $a > 1$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Show that C intersects the x-axis twice. [1]

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(iii) Justifying your answer, find the number of stationary points on C. [2]

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(iv) Sketch C, stating the coordinates of its point of intersection with the y-axis. [3]

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Q16

9 The curve C has equation

$$y = \frac{5x^2 + 5x + 1}{x^2 + x + 1}.$$

(i) Find the equation of the asymptote of C. [2]

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(ii) Show that, for all real values of x, $-\frac{1}{3} \leqslant y < 5$. [4]

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(iii) Find the coordinates of any stationary points of C. [2]

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(iv) Sketch C, stating the coordinates of any intersections with the y-axis. [2]

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Q17

9 The curve C has equation $y = \dfrac{x^2 - 3x + 6}{1 - x}$.

(i) Find the equations of the asymptotes of C. [3]

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(ii) Find the coordinates of the turning points of C. [3]

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(iii) Find the coordinates of any intersections with the coordinate axes. [2]

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(iv) Sketch C. [3]

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Q18

9 The curve C has equation

$$y = \frac{3x - 9}{(x - 2)(x + 1)}.$$

(i) Find the equations of the asymptotes of C. [2]

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(ii) Show that there is no point on C for which $\frac{1}{3} < y < 3$. [4]

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(iii) Find the coordinates of the turning points of C. [3]

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(iv) Sketch C. [3]

Craftear

Q19

7 A curve C has equation $y = \dfrac{x^2}{x-2}$. Find the equations of the asymptotes of C. [3]

Show that there are no points on C for which $0 < y < 8$. [4]

Sketch C, giving the coordinates of the turning points. [3]

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[Turn over

Q20

5 The curve C has equation $y = \dfrac{x+2}{x^2-9}$. Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} < 0$ at all points on C. [3]

State the equations of the asymptotes of C. [2]

Sketch C, showing the coordinates of any points of intersection with the coordinate axes. [3]

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[Turn over

Q21

10 The curve C has equation $y = \dfrac{4x^2 - 3x}{x^2 + 1}$. Verify that the equation of C may be written in the form $y = -\dfrac{1}{2} + \dfrac{(3x-1)^2}{2(x^2+1)}$ and also in the form $y = \dfrac{9}{2} - \dfrac{(x+3)^2}{2(x^2+1)}$. [3]

Hence show that $-\frac{1}{2} \leqslant y \leqslant \frac{9}{2}$. [2]

Without differentiating, write down the coordinates of the turning points of C. [2]

State the equation of the asymptote of C. [1]

Sketch the graph of C, stating the coordinates of the intersections with the coordinate axes and the asymptote. [3]

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[Turn over

Q22

8 The curve C has equation $y = \dfrac{2x^2 + kx}{x + 1}$, where k is a constant. Find the set of values of k for which C has no stationary points. [5]

For the case $k = 4$, find the equations of the asymptotes of C and sketch C, indicating the coordinates of the points where C intersects the coordinate axes. [6]

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[Turn over

12 **OR**

The curve C has equation

$$y = \frac{ax^2 + bx + c}{x + d},$$

where a, b, c and d are constants. The curve cuts the y-axis at $(0, -2)$ and has asymptotes $x = 2$ and $y = x + 1$.

(i) Write down the value of d. [1]

(ii) Determine the values of a, b and c. [6]

(iii) Show that, at all points on C, either $y \leqslant 3 - 2\sqrt{6}$ or $y \geqslant 3 + 2\sqrt{6}$. [7]

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11: **EITHER**

Express $\frac{2x^2 - x + 5}{x^2 - 1}$ in the form $2 + \frac{A}{x - 1} + \frac{B}{x + 1}$, where A and B are integers to be found. [3]

The curve C has equation $y = \frac{2x^2 - x + 5}{x^2 - 1}$. Show that there are two distinct values of x for which $\frac{dy}{dx} = 0$. [4]

Sketch C, stating the equations of the asymptotes and giving the coordinates of any points of intersection with the coordinate axes and with the asymptotes. You do not need to find the coordinates of the turning points. [7]

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Q25

4 A curve C has equation $y = \dfrac{2x^2 + x - 1}{x - 1}$. Find the equations of the asymptotes of C. [3]

Show that there is no point on C for which $1 < y < 9$. [4]

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Q26

10 The curve C has equation $y = \dfrac{2x^2 - 3x - 2}{x^2 - 2x + 1}$. State the equations of the asymptotes of C. [2]

Show that $y \leqslant \frac{25}{12}$ at all points of C. [4]

Find the coordinates of any stationary points of C. [3]

Sketch C, stating the coordinates of any intersections of C with the coordinate axes and the asymptotes. [4]

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[Turn over

10 The curve C has equation

$$y = \frac{px^2 + 4x + 1}{x + 1},$$

where p is a positive constant and $p \neq 3$.

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Find the value of p for which the x-axis is a tangent to C, and sketch C in this case. [4]

(iii) For the case $p = 1$, show that C has no turning points, and sketch C, giving the exact coordinates of the points of intersection of C with the x-axis. [5]

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7 The curve C has equation

$$y = \frac{2x^2 + 5x - 1}{x + 2}.$$

Find the equations of the asymptotes of C. [3]

Show that $\frac{dy}{dx} > 2$ at all points on C. [3]

Sketch C. [3]

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9 The curve C has equation

$$y = \frac{2x^2 + 2x + 3}{x^2 + 2}.$$

Show that, for all x, $1 \leqslant y \leqslant \frac{5}{2}$. [4]

Find the coordinates of the turning points on C. [3]

Find the equation of the asymptote of C. [2]

Sketch the graph of C, stating the coordinates of any intersections with the y-axis and the asymptote. [2]

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7 The curve C has equation

$$y = \lambda x + \frac{x}{x-2},$$

where λ is a non-zero constant. Find the equations of the asymptotes of C. [3]

Show that C has no turning points if $\lambda < 0$. [3]

Sketch C in the case $\lambda = -1$, stating the coordinates of the intersections with the axes. [3]

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Q31

9 The curve C has equation $y = \dfrac{x^2 - 3x + 3}{x - 2}$. Find the equations of the asymptotes of C. [3]

Show that there are no points on C for which $-1 < y < 3$. [4]

Find the coordinates of the turning points of C. [3]

Sketch C. [2]

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[Turn over

Q32

6 The curve C has equation $y = \dfrac{x^2}{x-2}$. Find the equations of the asymptotes of C. [3]

Find the coordinates of the turning points on C. [3]

Draw a sketch of C. [3]

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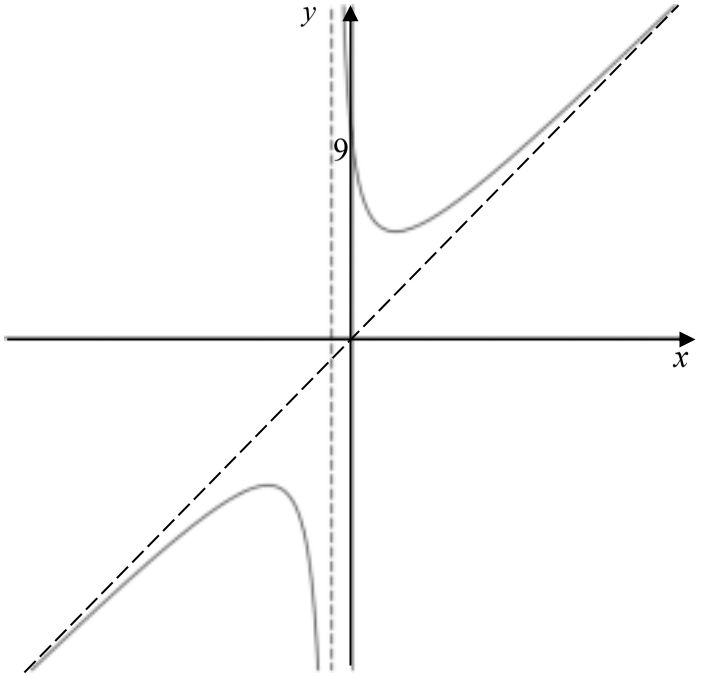
RATIONAL FUNCTIONS
MS

Craftear

Q1

Question	Answer	Marks	Guidance
7(a)	$x=-1$	**B1**	States vertical asymptote.
	$y=\dfrac{x(x+1)+9}{x+1}=x+\dfrac{9}{x+1}$	**M1**	Finds oblique asymptote.
	$y=x$	**A1**	
		3	
7(b)	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=1-9(x+1)^{-2}=0\Rightarrow(x+1)^2=9$	**M1 A1**	Differentiates and sets derivative equal to 0.
	$(2,5)$	**A1**	
	$(-4,-7)$	**A1**	
		4	
7(c)		**B1**	Axes labelled and correct asymptotes drawn.
		B1	Upper branch with $(0,9)$ stated or shown on diagram.
		B1	Lower branch correct and good approach to asymptotes throughout, no extra branches.
		3	

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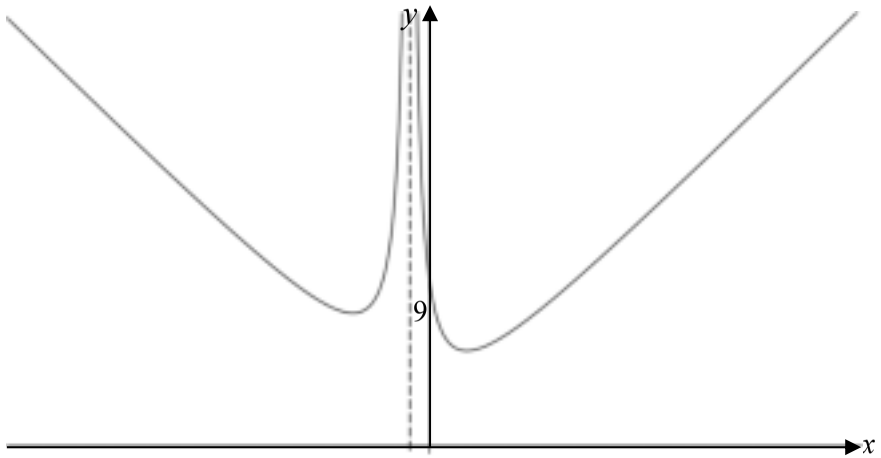
Question	Answer	Marks	Guidance
7(d)		**B1 FT**	FT from sketch in **(c)** with asymptotes shown.
	$x^2+x+9=\dfrac{13}{2}(x+1)$ or $x^2+x+9=-\dfrac{13}{2}(x+1)$ $x^2-\frac{11}{2}x+\frac{5}{2}=0$ or $x^2+\frac{15}{2}x+\frac{31}{2}=0$	**M1 M1**	Finds critical points, award M1 for each case. May state that $x^2+x+9=-\frac{13}{2}(x+1)$ has no real solutions since $7>\frac{13}{2}$.
	$x=\frac{1}{2}, 5$	**A1**	
	$x<\frac{1}{2}$ and $x>5$.	**A1**	
		5	

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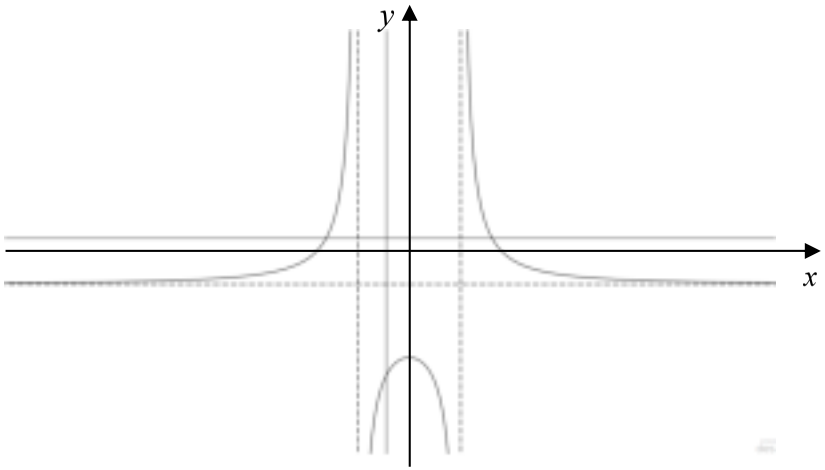
Q2

Question	Answer	Marks	Guidance
7(a)	$x=\frac{1-\sqrt{5}}{2}$, $x=\frac{1+\sqrt{5}}{2}$	**B1**	Vertical asymptotes. Must be exact.
	$y=-1$	**B1**	Horizontal asymptote.
		2	
7(b)	$\frac{dy}{dx}=\frac{(1+x-x^2)(2x-1)-(x^2-x-3)(1-2x)}{\left(1+x-x^2\right)^2}$	**M1**	Finds $\frac{dy}{dx}$.
	$(2x-1)(-2)=0$	**M1**	Sets their $\frac{dy}{dx}$ equal to 0 and forms equation.
	$\left(\frac{1}{2},-\frac{13}{5}\right)$	**A1**	WWW
		3	
7(c)		**B1**	Both axes labelled and correct asymptotes shown.
		B1	Correct shape and position, with all asymptotic behaviour clear.
	$\left(\frac{1}{2}+\frac{1}{2}\sqrt{13},0\right)$, $\left(\frac{1}{2}-\frac{1}{2}\sqrt{13},0\right)$, $(0,-3)$	**B1**	States exact coordinates of intersections with axes.
		3	

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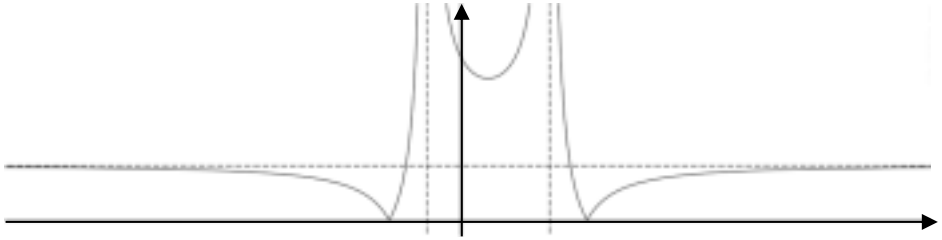
7(d)		**B1 FT**	FT from sketch in **(c).**
		B1	Correct shape as x tends to infinity and intersections with x axis.
	$\frac{x^2-x-3}{1+x-x^2}=3$ or $\frac{x^2-x-3}{1+x-x^2}=-3$ $4x^2-4x-6=0$ or $-2x^2+2x=0$	**M2**	Finds critical points, award M1 for each case.
	$x=\frac{1}{2}+\frac{1}{2}\sqrt{7},\quad x=\frac{1}{2}-\frac{1}{2}\sqrt{7},\ x=0,\quad x=1$	**A1**	Must be exact.
	$x<\frac{1}{2}-\frac{1}{2}\sqrt{7},\ 0<x<1,\ x>\frac{1}{2}+\frac{1}{2}\sqrt{7}$	**A1 FT**	Must be three distinct regions and strict inequalities.
		6	

Q3

Question	Answer	Marks	Guidance
7(a)	$x=1,\ x=-1$	**B1**	Vertical asymptotes.
	$y=0$	**B1**	Horizontal asymptote.
		2	
7(b)	$\frac{dy}{dx}=\frac{(4-4x^2)(4)-(4x+5)(-8x)}{\left(4-4x^2\right)^2}$	**M1**	Finds $\frac{dy}{dx}$.
	$16x^2+40x+16=0$	**M1**	Sets equal to 0 and forms equation.
	$\left(-2,\frac{1}{4}\right),\quad \left(-\frac{1}{2},1\right)$	**A1 A1**	
		4	

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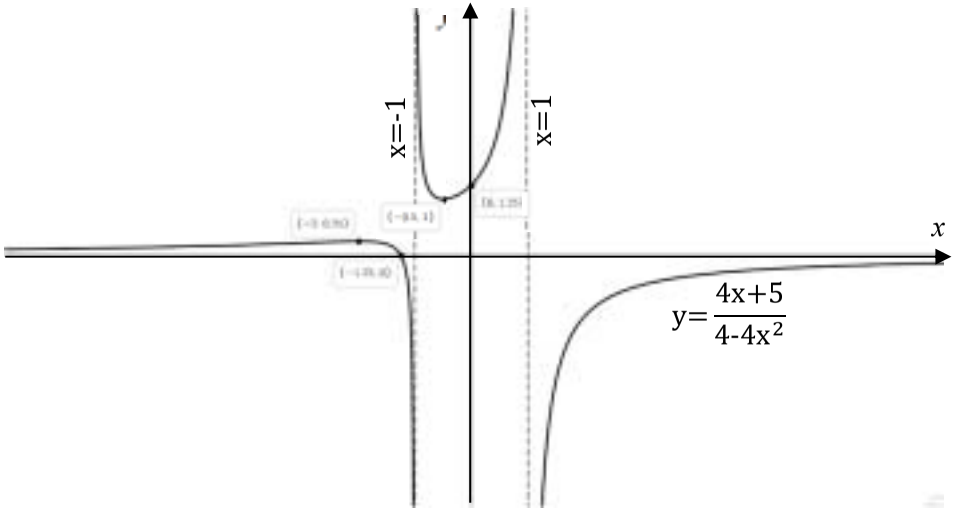
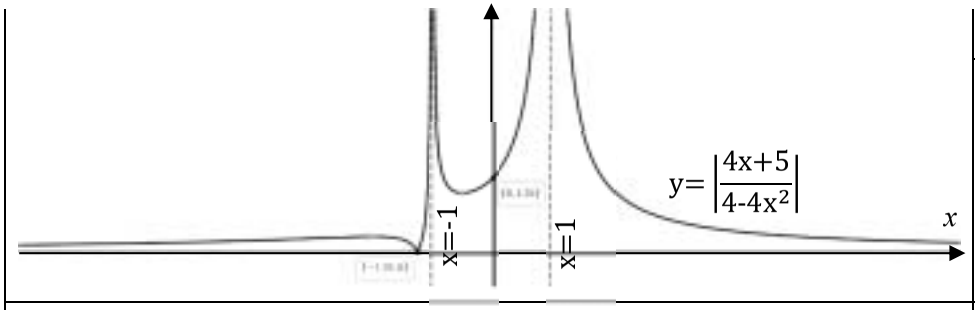
7(c)		**B1**	Axes and asymptotes.
		B1	Correct shape and position.
	$(-\frac{5}{4},0)$, $\left(0,\frac{5}{4}\right)$	**B1**	States coordinates of intersections with axes.
		3	
7(d)		**B1 FT**	FT from sketch in part **(c).**
		B1	Correct shape at infinity.
	$\frac{4x+5}{4-4x^2}=\frac{5}{4}$ or $\frac{4x+5}{4-4x^2}=-\frac{5}{4}$ $5x^2+4x=0$ or $5x^2-4x-10=0$	**M2**	Finds critical points, award M1 for each case.
	$x=-\frac{4}{5}$, $x=0$ or $x=\frac{2}{5}-\frac{3}{5}\sqrt{6}$, $x=\frac{2}{5}+\frac{3}{5}\sqrt{6}$	**A1**	A0 if –1.07, 1.87
	$\frac{2}{5}-\frac{3}{5}\sqrt{6}<x<-\frac{4}{5}$, $0<x<\frac{2}{5}+\frac{3}{5}\sqrt{6}$, $x\neq\pm1$	**A1 FT**	Condone exclusion of $x=\pm1$ from the range.
		6	

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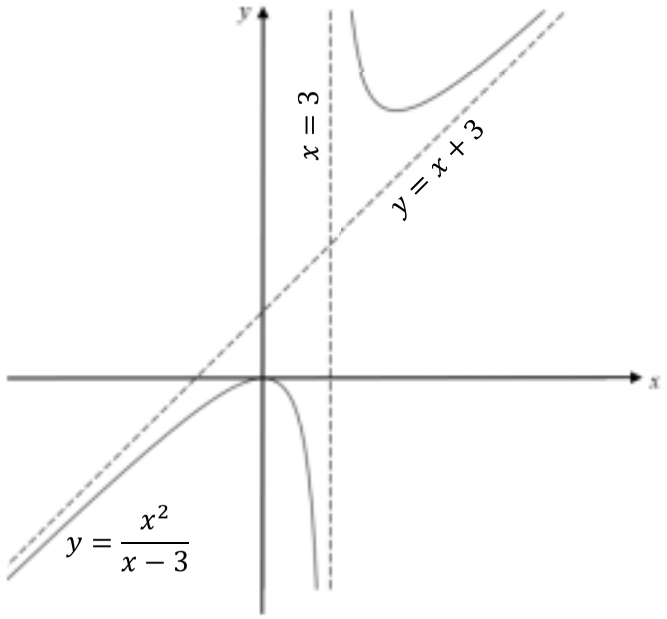
Q4

Question	Answer	Marks	Guidance
6(a)	$x = 3$	**B1**	States vertical asymptote.
	$y = x + 3 + \dfrac{9}{x-3}$ leading to $y = x + 3$	**M1 A1**	Finds oblique asymptote.
		3	
6(b)	$yx - 3y = x^2$ leading to $x^2 - yx + 3y = 0$	**M1 A1**	Forms quadratic in x.
	$y^2 - 4(3y) < 0$ leading to $y^2 - 12y < 0$	**M1**	Uses that discriminant is negative.
	$0 < y < 12$	**A1**	AG
		4	
6(c)		**B1**	Axes and asymptotes.
		B1	Branches correct.
		2	

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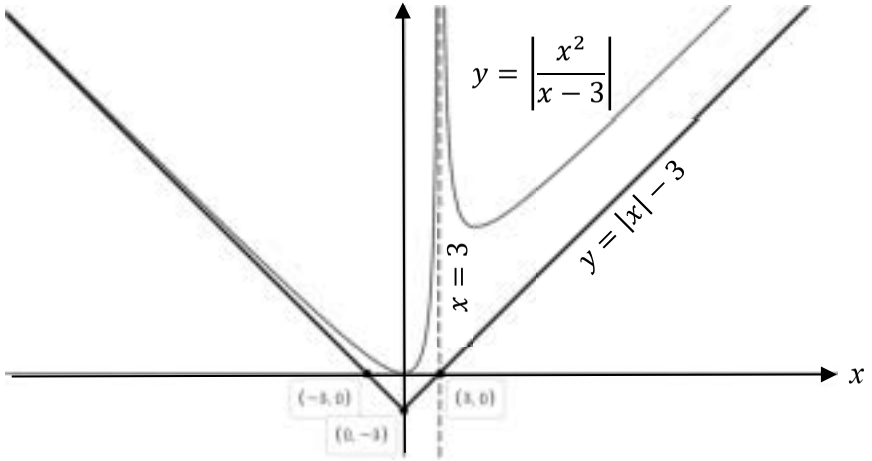
6(d)(i)		**B1 FT**	FT from sketch in **(c)**.
		B1	Correct shape at infinity.
		B1	Correct shape of $y = \lvert x \rvert - 3$.
		B1	Correct intercepts with axes (may be seen on graph).
		4	
6(d)(ii)	$c \leqslant -3$	**B1**	
		1	

Q5

Question	Answer	Marks	Guidance
6(a)	$x = 1$	**B1**	States vertical asymptote.
	$x^2 + x - 1 = (x-1)(x+2) + 1 \Rightarrow y = x + 2$	**M1 A1**	Finds oblique asymptote.
		3	
6(b)	$yx - y = x^2 + x - 1 \Rightarrow x^2 + (1-y)x + y - 1 = 0$	**M1 A1**	Forms quadratic in x.
	$(1-y)^2 - 4(y-1) < 0 \Rightarrow y^2 - 6y + 5 < 0$	**M1**	Uses that discriminant is negative if there are no values of x.
	$1 < y < 5$	**A1**	AG
		4	

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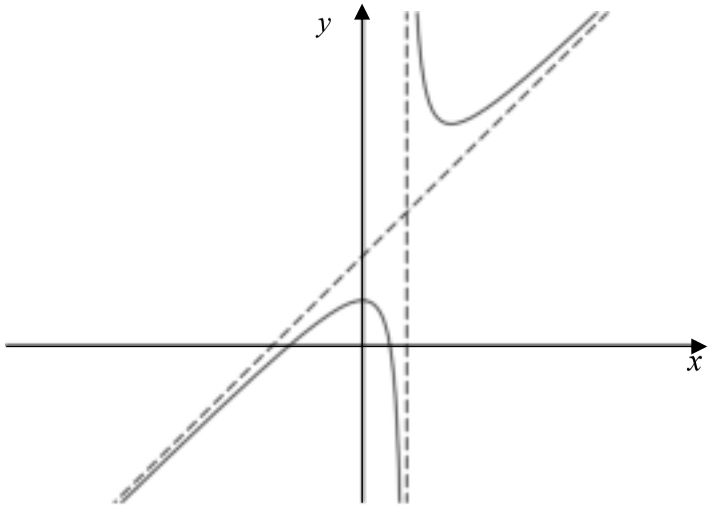
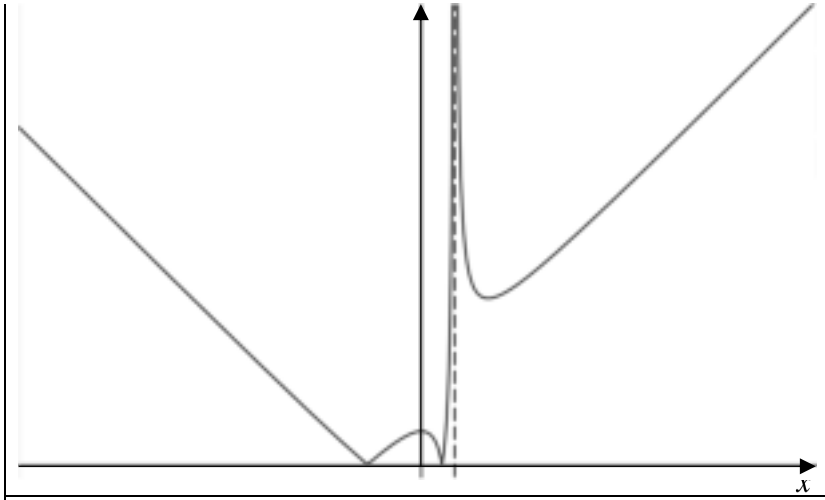
6(c)		**B1**	Axes and asymptotes.
		B1	Branches correct.
	$(0,1),\quad \left(-\frac{1}{2}+\frac{1}{2}\sqrt{5},0\right),\quad \left(-\frac{1}{2}-\frac{1}{2}\sqrt{5},0\right)$	**B1**	States coordinates of intersections with axes, can be labelled on graph. Accept $(0.618,0)$ and $(-1.62,0)$.
		3	
6(d)		**B1 FT**	FT from sketch in **(c)** with both branches.
		B1	Correct shape at extremities.
		2	

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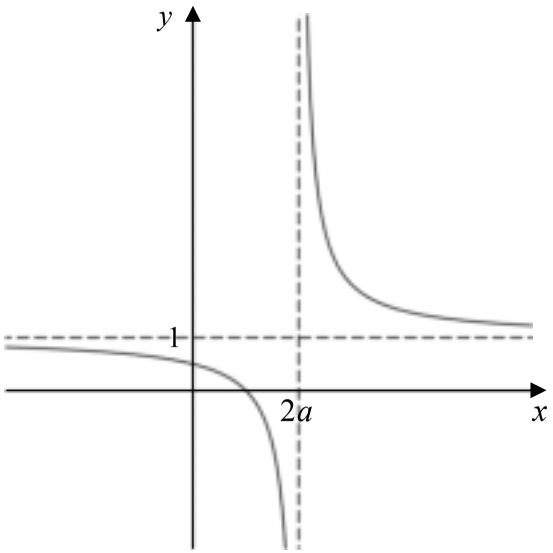
Q6

Question	Answer	Marks	Guidance
6(a)		**B1**	Asymptotes labelled.
		B1	Correct position and shape. Not too truncated.
		2	
6(b)(i)	$\frac{dy}{dx} = -\frac{2a(x-a)}{(x-2a)^3} = 0$	**M1**	Sets $\frac{dy}{dx} = 0$.
	Solves for x.	**M1**	
	Since $x \neq 2a$, stationary point is $(a, 0)$.	**A1**	
		3	
6(b)(ii)	$(x-a)^2 = (x-a)(x-2a)$ or $(x-a)^2 = -(x-a)(x-2a)$	**M1**	Finds a critical point.
	$x = a$ and $\frac{3}{2}a$ **or** $(a, 0)$ **or** $(\frac{3}{2}a, 1)$	**A1**	Both x values or one correct point.
	$(a, 0)$ and $(\frac{3}{2}a, 1)$	**A1**	
		3	

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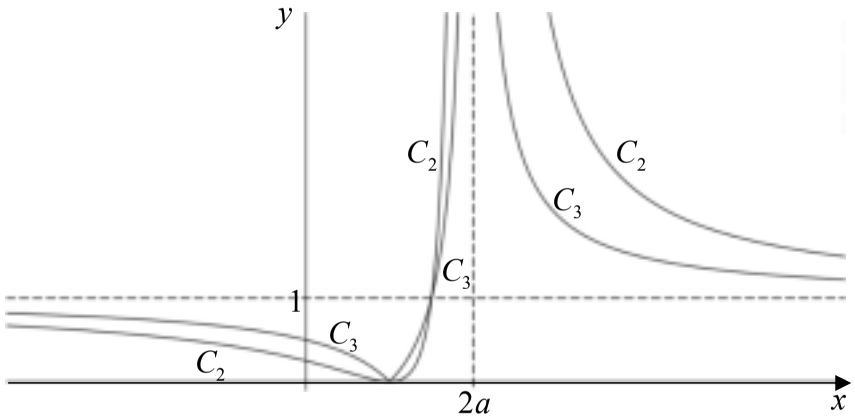
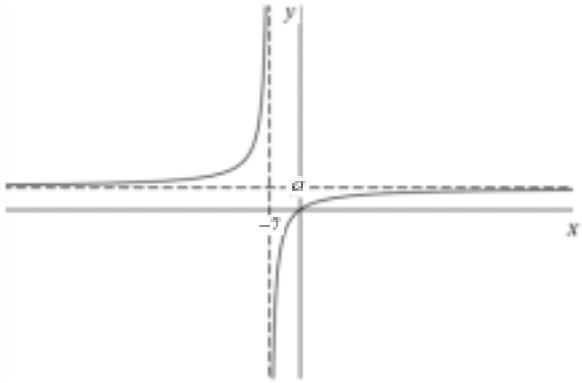
6(c)		**B1**	Axes and asymptotes.
		B1 FT	C_3 FT from their sketch in part **(a)**. Clearly identified.
		B1	Correct shape of C_2. Clearly identified.
		B1	Relative positions and intersections correct.
	$x \leqslant \frac{3}{2}a$	**B1**	Accept algebraic method.
		5	

Q7

Question	Answer	Marks
1(a)		**B1**
	(**B1** for axes and asymptotes correct, **B1** for branches correct)	**B1**
		2

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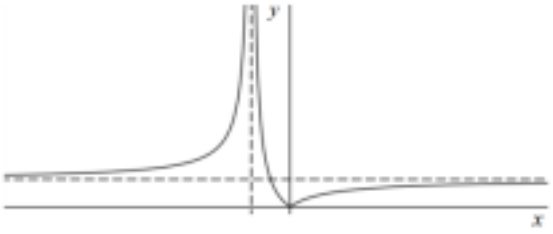
1(b)	 (**B1 FT** from sketch in part **(a)**)	**B1 FT**
	$\frac{ax}{x+7}=\frac{a}{2}$ or $\frac{ax}{x+7}=-\frac{a}{2}$	**M1**
	$x=7$ and $x=-\frac{7}{3}$	**A1**
	$x<-7$, $-7<x<-\frac{7}{3}$, $x>7$	**A1**
		4

Q8

Question	Answer	Marks
3(a)	$x=-\frac{1}{2}$	**B1**
	$y=\frac{(2x+1)(\frac{1}{2}x-\frac{1}{4})+\frac{1}{4}}{2x+1}=\frac{1}{2}x-\frac{1}{4}+\frac{1}{4(2x+1)}$	**M1**
	$y=\frac{1}{2}x-\frac{1}{4}$	**A1**
		3
3(b)	$\frac{dy}{dx}=\frac{1}{2}-\frac{1}{2(2x+1)^2}=0\Rightarrow(2x+1)^2=1$	**M1**
	$x=0,-1$	**A1**
	$(0,0),(-1,-1)$	**A1**
		3

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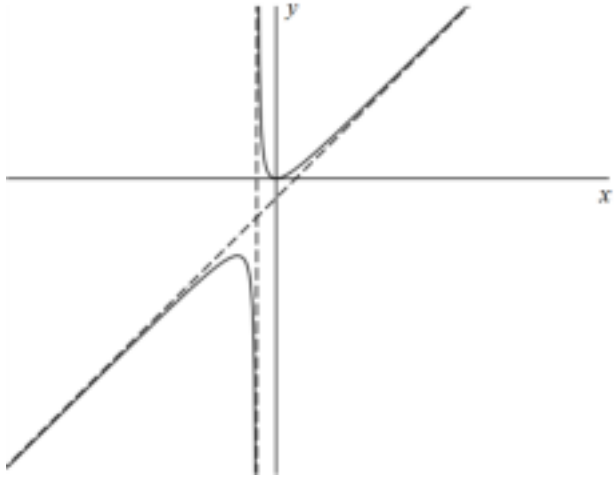
3(c)		**B1**
		B1
	(**B1** for axes and asymptotes correct, **B1** for upper branch correct, **B1** for lower branch correct)	**B1**
		3

Q9

Question	Answer	Marks
6(a)	$x=\frac{3}{2}$	**B1**
	$-2x^2+x+10=(2x-3)(-x-1)+7 \Rightarrow y=-x-1$	**M1 A1**
		3
6(b)	$\frac{dy}{dx}=\frac{(2x-3)(1-4x)-2(10+x-2x^2)}{(2x-3)^2}$	**M1**
	$4x^2-12x+23=0 \quad \left(\text{or } \frac{dy}{dx}=-1-\frac{14}{(2x-3)^2}\right)$	**A1**
	$12^2-4(4)(23)=-224<0$ (or $y'<0$) $\Rightarrow$ No turning points.	**A1**
		3

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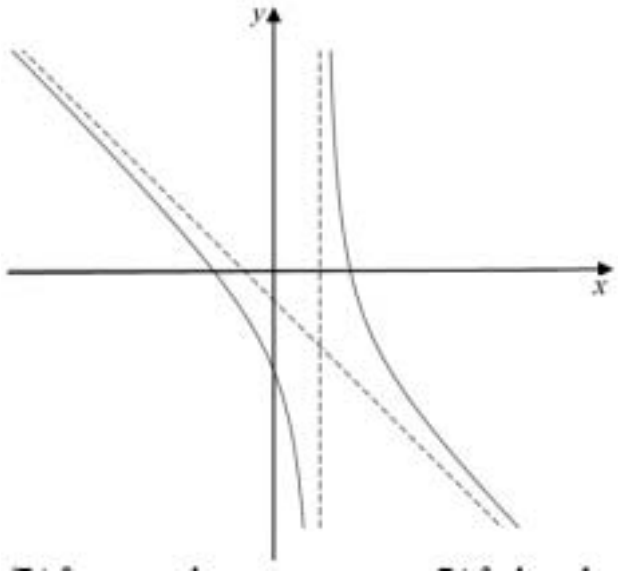
6(c)		**B1**
	(**B1** for axes and asymptotes correct, **B1** for branches correct)	**B1**
	$(0,-\frac{10}{3}),(-2,0),(\frac{5}{2},0)$	**B1**
		3

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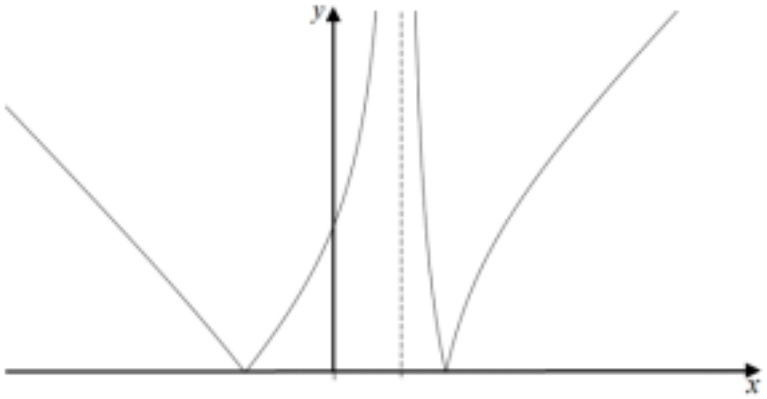
Question	Answer	Marks
6(d)		**B1 FT**
	(**B1 FT** for their sketch in **(c)**, **B1** for correct shape at infinity)	**B1**
	$\frac{10+x-2x^2}{2x-3}=4$ or $\frac{10+x-2x^2}{2x-3}=-4$ $2x^2+7x-22=0$ or $2x^2-9x+2=0$	**M2**
	$x=-\frac{11}{2}, 2$ or $x=\frac{1}{4}\left(9\pm\sqrt{65}\right)$	**A1**
	$-\frac{11}{2}<x<\frac{1}{4}\left(9-\sqrt{65}\right)$ and $2<x<\frac{1}{4}\left(9+\sqrt{65}\right)$	**A1 FT**
		6

Q10

Question	Answer	Marks	Guidance
4(i)	$x^2+1=(ax+b)(2x+1)+c$	**M1**	Uses that $2x+1$ is the quotient.
	$\Rightarrow a=\frac{1}{2}, b=-\frac{1}{4}$	**A1 A1**	
		3	
4(ii)	$x=\frac{1}{2}$	**B1 FT**	
		1	

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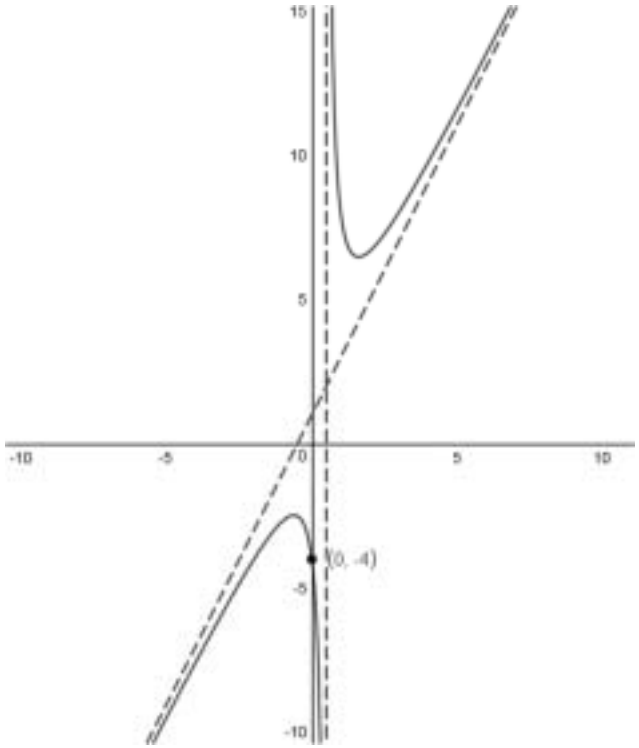
Question	Answer	Marks	Guidance
4(iii)		**B1**	Intersection (0,-4) given and asymptotes drawn.
		B1	Left branch correct.
		B1 FT	Right branch correct. Deduct at most one mark for poor forms at infinity.
		3	

Q11

Question	Answer	Marks	Guidance
10(i)	$x = -5$ and $y = a$	**B1 B1**	
		2	
10(ii)	$x^2 + (a+10)x + 5a + 26 = (x+5)(x+a+5) + 1$	**M1**	By inspection or long division.
	oblique asymptote is $y = x + a + 5$	**A1**	
		2	

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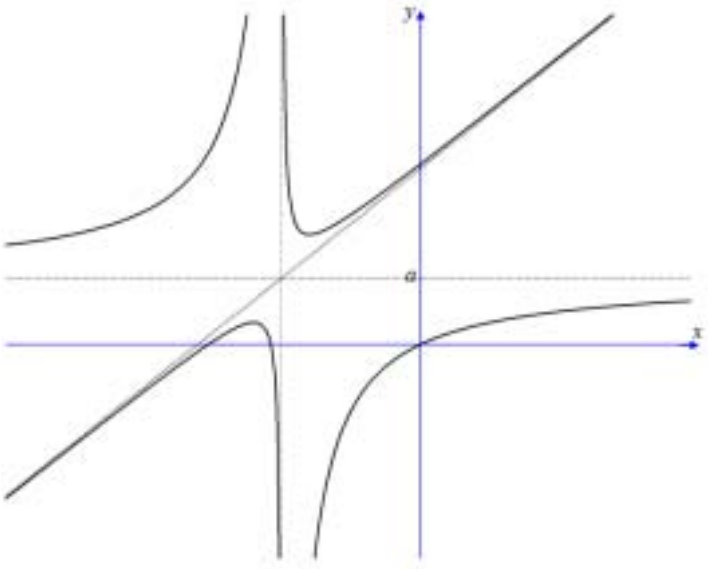
10(iii)	$x^2+10x+5a+26=0$	**M1**	Puts y–values equal and forms quadratic equation.
	$10^2-4(5a+26)=-4-20a<0$ so no intersection point	**A1**	Correct discriminant and conclusion.
		2	
10(iv)	$(x+5)(2x+a+10)-x^2-ax-10x-5a-26=0$	**M1**	Differentiates and forms quadratic equation.
	$x^2+10x+24=0$	**A1**	
	Stationary points are $(-4,a+2)$ and $(-6,a-2)$	**A1**	Must have both points.
		3	
10(v)		**B1**	Asymptotes drawn, intersection correct.
		B1	C_1 correct.
		B1	C_2 correct.
		3	

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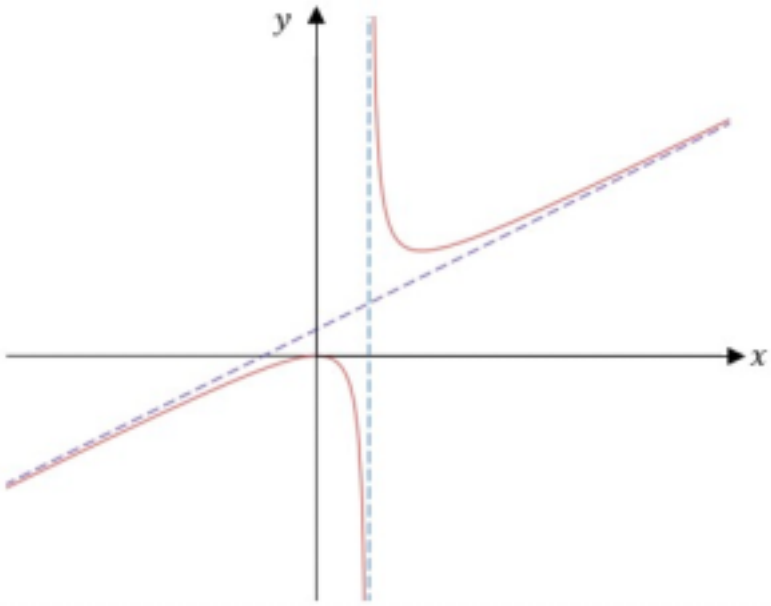
Q12

Question	Answer	Marks	Guidance
6(i)	$x=\frac{1}{k}$	**B1**	States vertical asymptote.
	$y=\frac{(kx-1)\left(k^{-1}x+k^{-2}\right)+k^{-2}}{kx-1}$	**M1**	Finds oblique asymptote.
	Oblique asymptote is $y=k^{-1}x+k^{-2}$	**A1**	
		3	
6(ii)	$y'=\frac{2x(kx-1)-kx^2}{(kx-1)^2}=0\Rightarrow kx^2-2x=0$	**M1**	Differentiates and equates to 0.
	$x=0,\ 2k^{-1}$	**A1**	Finds x-coordinates.
	$(0,0),\ \left(2k^{-1},4k^{-2}\right)$	**A1**	Finds y-coordinates
		3	
6(iii)		**B1**	Axes and asymptotes correct.
		B1	Upper branch correct.
		B1	Lower branch correct. Deduct at most 1 mark for poor forms at infinity.
		3	

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Q13

Question	Answer	Marks	Guidance
6(i)	Vertical asymptote is $x = -b$.	**B1**	
	$x^2 + b = (x+b)(x-b) + b^2 + b$ or $x+b \overline{)x^2 + 0x + b}$ $\quad x - b$	**M1**	By inspection or long division.
	Thus the oblique asymptote is $y = x - b$	**A1**	
		3	
6(ii)	If $y = 0$ then $x^2 + b = 0$ which has no real root.	**B1**	Must refer to b > 0 OE
		1	
6(iii)	$\frac{dy}{dx} = \frac{2x(x+b) - (x^2+b)}{(x+b)^2} = 0 \Rightarrow x^2 + 2bx - b = 0$ Or differentiating $y = x - b + \frac{b^2+b}{x+b}$ and setting $\frac{dy}{dx} = 0$ gives $1 - \frac{b^2+b}{(x+b)^2} = 0$.	**M1**	Find $\frac{dy}{dx}$ and set = 0
	$b^2 + b > 0$ Therefore there are two stationary points on C	**A1**	Use discriminant or $(x + b)^2$ to show two stationary points
		2	

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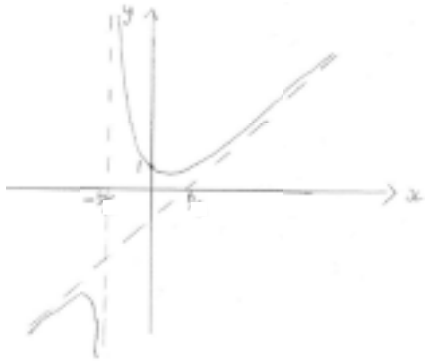
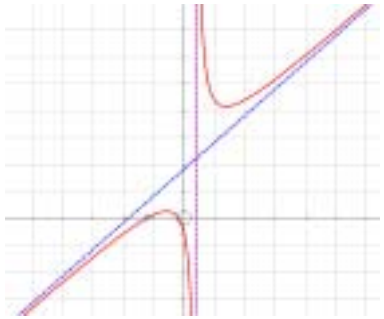
6(iv)		**B1**	Intersection (0,1) g ven and asymptotes drawn
		B1 B1	Each branch correct Penalise at most one mark for poor forms at infinity
		3	

Q14

Question	Answer	Marks	Guidance
4(i)	$(-6,0),(-1,0)$	**B1**	States points of intersection with x-axis.
	$(0,-3)$	**B1**	States y-intercept
4(ii)	One asymptote is $x=2$.	**B1**	
	$y=x+9+\dfrac{24}{x-2} \Rightarrow$ other asymptote is $y=x+9$.	**M1 A1**	By inspection or long division. A0 if error in division
4(iii)		**B1**	Sketches axes and asymptotes, labelled or to scale
		B1	Upper branch correctly located and orientated.
		B1	Lower branch correctly located and orientated. Penalise at most 1 mark for poor forms at infinity
		8	

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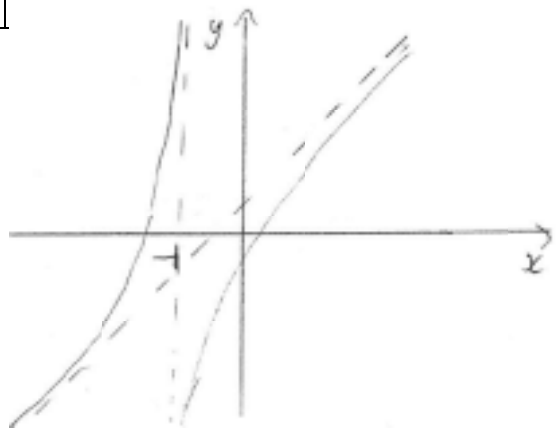
Q15

Question	Answer	Marks	Guidance
6(i)	Vertical asymptote is $x=-1$.	**B1**	
	$x^2+ax-1=(x+1)(x+a-1)-a$ or $x+1\overline{)x^2+ax-1}$ $\quad x+a-1$	**M1**	By inspection or long division.
	Thus the oblique asymptote is $y=x+a-1$	**A1**	
		3	
6(ii)	$a^2+4>0$	**B1**	
		1	
6(iii)	$\dfrac{(x+1)(2x+a)-(x^2+ax-1)}{(x+1)^2}=0\Rightarrow x^2+2x+a+1=0$ Discriminant $=4-4(a+1)<0$	**M1**	
	Therefore there are no stationary points on C.	**A1**	
		2	
6(iv)		**B1**	Correct y-intercept and asymptotes drawn.
		B1 B1	Each branch correct
		3	

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Q16

Question	Answer	Marks	Guidance
9(i)	$y = 5 - \dfrac{4}{x^2 + x + 1}$	**M1**	Alt method: Finding limit
	As $x \to \pm\infty$, $y \to 5 \therefore y = 5$ CAO	**A1**	
		2	
9(ii)	$yx^2 + yx + y = 5x^2 + 5x + 1$ $\Rightarrow (y-5)x^2 + (y-5)x + (y-1) = 0$	**B1**	Forms quadratic equation in x.
	For real x, $(y-5)^2 - 4(y-5)(y-1) \geqslant 0$ (condone >)	**M1**	Uses discriminant
	$\Rightarrow (y-5)(3y+1) \leqslant 0$	**M1**	Factorising
	$\Rightarrow -\dfrac{1}{3} \leqslant y < 5$, because $y = 5$ is an asymptote (www)	**A1**	Explaining strict upper inequality (AG)
		4	
9(iii)	$y' = 0 \Rightarrow (x^2 + x + 1)(10x + 5) - (5x^2 + 5x + 1)(2x + 1) = 0$	**M1**	Differentiates and equates to 0.
	$\Rightarrow 4(2x + 1) = 0 \Rightarrow x = -\dfrac{1}{2}, y = -\dfrac{1}{3}$	**A1**	
		2	

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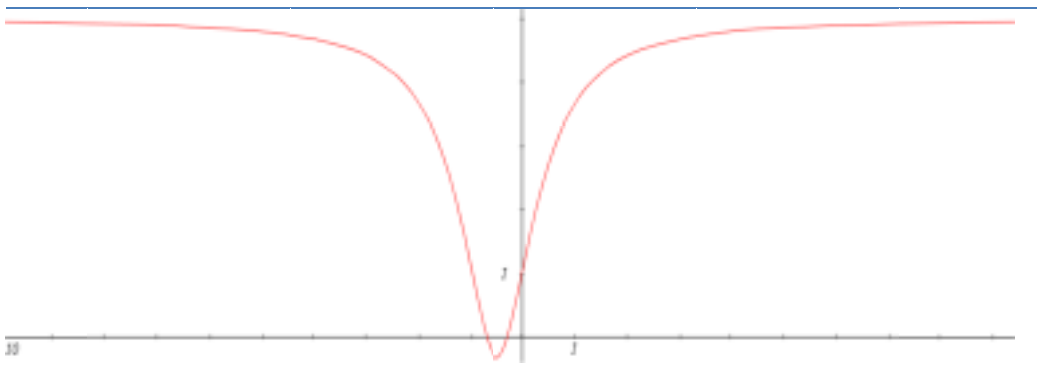
9(iv)		**B1FT**	Positive y-intercept at (0,1), FT dep on minimum point from **(iii)**.
		B1	Correct asymptote and completely correct graph.
		2	

Q17

Question	Answer	Marks	Guidance
9(i)	$x=1$	**B1**	
	$y=2-x$	**M1A1**	
	Total:	**3**	
9(ii)	$y'=-1+4(1-x)^{-2}=0$	**M1**	
	$\Rightarrow x=-1,3$	**A1**	
	Turning points are $(-1,5)$ and $(3,-3)$	**A1**	
	Total:	**3**	
9(iii)	$(0\ ,\ 6)$	**B1**	
	$y=0\Rightarrow x^2-3x+6\ ;\ \Delta=9-24\Rightarrow$ No intersection with x-axis	**B1**	
	Total:	**2**	

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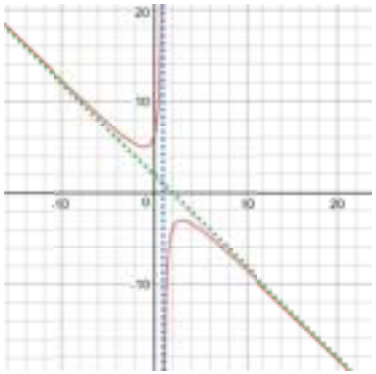
9(iv)		**B1 FT**	Asymptotes correct.
		B1B1	Each branch.
		Total:	**3**

Q18

Question	Answer	Marks	Guidance
9(i)	Degree of numerator < degree of denominator ⇒ $y = 0$ is horizontal asymptote.	**B1**	
	$(x+1)(x-2)=0 \Rightarrow x=-1$ and $\Rightarrow x=2$ are vertical asymptotes.	**B1**	
		2	
9(ii)	$yx^2-(y+3)x+9-2y=0$	**M1**	
	No points on C if $(y+3)^2-4y(9-2y)<0$	**M1**	
	$\Rightarrow 9y^2-30y+9<0 \Rightarrow 3y^2-10y+3<0$	**A1**	
	$\Rightarrow (3y-1)(y-3)<0 \Rightarrow \frac{1}{3}<y<3$	**A1**	AG
		4	

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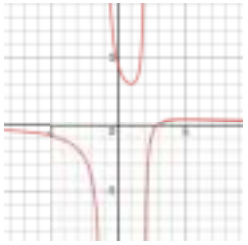
9(iii)	$\frac{dy}{dx} = 0 \Rightarrow 3\left(x^2 - x - 2\right) - (3x - 9)(2x - 1) = 0$	**B1**	
	$\Rightarrow \ldots \Rightarrow (x-1)(x-5) = 0$	**B1**	
	$\Rightarrow$ Turning points are (1,3) and $\left(5, \frac{1}{3}\right)$.	**B1**	
		3	
9(iv)	 Axes, asymptotes and points on axes (0, 4.5) (3,0).	**B1**	
	RH branch ; Other two branches	**B1B1**	
		3	

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Q19

7	Vertical asymptote is $x = 2$. $y = x + 2 + \dfrac{4}{x-2} \Rightarrow$ Oblique asymptote is $y = x + 2$. $y = \dfrac{x^2}{x-2} \Rightarrow x^2 - yx + 2y = 0$ Quadratic has no real roots (i.e. no points on C) if $\Delta < 0 \Rightarrow y^2 - 8y < 0$ $\Rightarrow y(y-8) < 0 \Rightarrow 0 < y < 8$.(**AG**) Correct inequality Axes and asymptotes. Each branch, showing (0,0) and (4,8). (Deduct at most 1 mark for poor forms at infinity and/or missing coordinates.)	**B1** **M1A1** [3] **B1** **M1** **M1A1** [4] **B1**√ **B1B1** [3]

Q20

5	$\dfrac{dy}{dx} = \dfrac{(x^2-9) - 2x(x+2)}{(x^2-9)^2}$ $= \dfrac{-x^2-4x-9}{(x^2-9)^2} = \dfrac{-(x+2)^2-5}{(x^2-9)^2} \Rightarrow \dfrac{dy}{dx} < 0$ Asymptotes: $x = \pm 3$; $y = 0$ Sketch: Axes and asymptotes ; Outside branches Middle branch, showing $\left(0, -\dfrac{2}{9}\right)$ and $(-2, 0)$. (Deduct at most 1 mark for poor forms at infinity.)	**B1** **M1A1** [3] **B1B1** [2] **B1B1** **B1** [3]

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Q21

10	$-\dfrac{1}{2} + \dfrac{(3x-1)^2}{2(x^2+1)} = \dfrac{-x^2-1+9x^2-6x+1}{2(x^2+1)} = \dfrac{8x^2-6x}{2(x^2+1)} = \dfrac{4x^2-3x}{x^2+1}$ $\dfrac{9}{2} - \dfrac{(x+3)^2}{2(x^2+1)} = \dfrac{9x^2+9-x^2-6x-9}{2(x^2+1)} = \dfrac{8x^2-6x}{2(\quad 1)} = \dfrac{4x^2-3x}{x^2+1}$ $\dfrac{(3x-1)^2}{2(x^2+1)} \geqslant 0 \Rightarrow y \geqslant -\dfrac{1}{2}$ $\dfrac{(x+3)^2}{2(x^2+1)} \geqslant 0 \Rightarrow y \leqslant \dfrac{9}{2} \Rightarrow -\dfrac{1}{2} \leqslant y \leqslant \dfrac{9}{2}$ (Not hence gets B1 only.) Turning points occur at equality, so $\left(-3, \dfrac{9}{2}\right)$ and $\left(\dfrac{1}{3}, -\dfrac{1}{2}\right)$ Asymptote is $y = 4$. Intersection with axes at (0, 0) and $\left(\dfrac{3}{4}, 0\right)$, Intersection with asymptote at $\left(-\dfrac{4}{3}, 4\right)$ Correct graph.	M1A1 A1 (3) B1 B1 (2) B1B1 (2) B1 (1) B1B1 B1 (3) **Total 11**

Q22

8	$y' = 0 \Rightarrow (x+1)(4x+k) - (2x^2 + kx) \times 1 = 0$	**M1**
	$\Rightarrow 4x^2 + (4+k)x + k - 2x^2 - kx = 0 \Rightarrow 2x^2 + 4x + k = 0$	**A1**
	$B^2 - 4AC < 0 \Rightarrow$ no stationary points $\Rightarrow 16 - 8k < 0$	**M1A1**
	$\Rightarrow k > 2$ for no stationary points.	**A1**
		[5]
	When $k = 4$:	
	Vertical asymptote: $x = -1$	**B1**
	Oblique asymptote: $y = 2x + 2 - \dfrac{2}{x+1} \Rightarrow y = 2x + 2$	**M1A1**
	Axes and asymptotes	**B1**
	Each branch.	**B1B1**
		[6]
		Total 11

Q23

12O (i)	$d = -2$	B1
		[1]
(ii)	$x = 0 \Rightarrow y = -2 = \dfrac{c}{d} \Rightarrow c = 4$	M1A1
	$y = ax + (b + 2a) + \dfrac{4 + 4a + 2b}{x - 2}$	M1A1
	Oblique asymptote $y = x + 1 \Rightarrow a = 1$, $b + 2a = 1 \Rightarrow b = -1$	A1A1
		[6]
(iii)	$y = \dfrac{x^2 - x + 4}{x - 2} \Rightarrow x^2 - (y+1)x + 2y + 4 = 0$	M1
	For real x $B^2 - 4AC > 0 \Rightarrow (y+1)^2 - 4(2y+4) > 0$	M1
	$\Rightarrow y^2 - y - 15 > 0$	A1
	$\Rightarrow (y-3)^2 > 24$ (or for solving $y^2 - 6y - 15 = 0$)	M1A1
	$\Rightarrow y - 3 < -2\sqrt{6}$ or $y - 3 > -2\sqrt{6}$ (or for thumbnail sketch)	M1
	$\Rightarrow y < 3 - 2\sqrt{6}$ or $y > 3 + 2\sqrt{6}$ (AG)	A1
	(Note: if done by differentiation, max and min must be demonstrated for full marks.)	[7]

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Q24

11	**EITH**	
	$2 + \dfrac{A}{x-1} + \dfrac{B}{x+1} = \dfrac{2(x^2-1) + A(x+1) + B(x-1)}{x^2-1} = \dfrac{2x^2 + (A+B)x + A - B - 2}{x^2-1} = \dfrac{2x^2 - x + 5}{x^2-1}$	M1
	$\Rightarrow A = 3$, $B = -4$	A1A1
		[3]
	$y' = 0 \Rightarrow (x^2 - 1)(4x - 1) - (2x^2 - x + 5).2x = 0 \Rightarrow x^2 - 14x + 1 = 0$	M1A1
	$B^2 - 4AC = (-14)^2 - 4 \times 1 \times 1 = 192 > 0 \Rightarrow$ two distinct turning points.	M1A1
		[4]
	Asymptotes are $x = 1$, $x = -1$: $y = 2$	B1B1
	$y = 2 \Rightarrow 2x^2 - x + 5 = 2x^2 - 2 \Rightarrow x = 7 \Rightarrow (7, 2)$ (Accept if labelled on graph.)	M1A1
	Sketch: Middle branch crossing y-axis at $(0, -5)$ and left branch.	B1
	Right branch.	B1
	Working to show no intersections with x-axis.	B1
		[7]

Q25

4	Vertical asymptote is $x = 1$ $y = 2x + 3 + 2(x-1)^{-1} \Rightarrow y = 2x + 3$ is the oblique asymptote. $2x^2 + (1-y)x + (y-1) = 0$ has real roots $\Leftrightarrow (1-y)^2 - 8(y-1) \geqslant 0$ $\Leftrightarrow y^2 - 10y + 9 \geqslant 0$ $\Leftrightarrow (y-1)(y-9) \geqslant 0$ Hence $(y-1)(y-9) < 0 \Rightarrow$ no real roots. i.e. $1 < y < 9 \Rightarrow$ no points on C. (AG) Thumbnail sketch, or similar, required.	B1 M1A1 (3) M1A1 M1 A1 (4) **[7]**

Q26

10	States asymptotes.	Vertical: $x = 1$ and Horizontal: $y = 2$	B1B1	2	
	Obtains quadratic form in x.	$yx^2 - 2yx + y = 2x^2 - 3x - 2$ $\Rightarrow (y-2)x^2 - (2y-3)x + (y+2) = 0$	M1A1		
	Uses $B^2 - 4AC \geq 0$ for real roots.	For real x $(2y-3)^2 - 4(y-2)(y+2) \geq 0$ $\Rightarrow 12y \leq 25 \Rightarrow y \leq \frac{25}{12}$.	M1 A1	4	
	Finds condition for $y' = 0$.	$y' = 0 \Rightarrow$ $(x^2 - 2x + 1)(4x - 3) - (2x^2 - 3x - 2)(2x - 2) = 0$	M1		
	Solves	$\Rightarrow x^2 - 8x + 7 = 0 \Rightarrow (x-7)(x-1) = 0$ $\Rightarrow x = 7$, (since $x = 1$ is vertical asymptote).	A1		
	Obtains stationary point.	Stationary point is $\left(7, \frac{25}{12}\right)$	A1		
	Sketch showing:	Axes and asymptotes (-0.5,0) , (2,0) , (0,–2) and (4,2) Left hand branch. Right hand branch.	B1 B1 B1 B1	4	**[13]**

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Q27

10 (i)	Vertical asymptote. Oblique asymptote.	Asymptotes : $x = -1$ $y = px + 4 - p + (p-3)(x+1)^{-1} \Rightarrow y = px + 4 - p$	B1 M1A1	3	
(ii)	Obtains value of p. Sketches curve.	$p = 4 \Rightarrow$ x-axis is a tangent Correct location of turning points and asymptotes. Each branch.	B1 B1 B1B1	4	
(iii)	Proves required result. Sketches graph.	$p = 1 \Rightarrow y = x + 3 - 2(x+1)^{-1} \Rightarrow y' = 1 + 2(x+1)^{-2}$ $(\geqslant 1)$ Intersections on x-axis at $\left(-2 \pm \sqrt{3}, 0\right)$ Each branch.	M1A1 B1 B1B1	5	**[12]**

Q28

7	States vertical asymptote	Vertical asymptote is $x = -2$	B1		
	Expresses y in a suitable form	$y = 2x + 1 - 3(x+2)^{-1}$	M1		
	States oblique asymptote	Oblique asymptote is $y = 2x + 1$	A1	**(3)**	
	Differentiates wrt x	$y' = 2 + 3(x+2)^{-2}$	M1A1		
	and explains result	$(x+2)^{-2} > 0 \Rightarrow y' > 2$ (AG)	A1	**(3)**	
	Sketches graph.	Axes and asymptotes	B1		
	(Deduct 1 mark for poor forms at infinity)	Each branch, showing intersection with axes.	B1B1	**(3)**	**[9]**

Q29

9(i) and (ii)	**Possible approach for first two parts together.**	Writes $y = \dfrac{2x^2+2x+3}{x^2+2} = 1 + \dfrac{(x+1)^2}{x^2+2}$	(B1)		
		States $\dfrac{(x+1)^2}{x^2+2} \geq 0 \Rightarrow y \geq 1$	(B1)		
		From this it is clear that (–1, 1) is a turning point.	(M1A1)		
		Writes $y = \dfrac{2x^2+2x+3}{x^2+2} = \dfrac{5}{2} - \dfrac{(x-2)^2}{2(x^2+2)}$	(B1)		
		States $\dfrac{(x-2)^2}{2(x^2+2)} \geq 0 \Rightarrow y \leq \dfrac{5}{2}$			
		From this it is clear that $(2, 2\frac{1}{2})$ is the other turning point.	(B1) (A1)	(7)	
	(i) can come after finding turning points: Continuous function (implied by graph)		(M1) (M1A1)		
	⇒ (2,2.5) Max and (–1,1) Min $\Rightarrow 1 \leq y \leq \dfrac{5}{2}$ (AG)		(A1)	(4)	
	N.B. Award B1 if Max and Min assumed without proof. i.e. 1/4 .				

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OR

9	Forms quadratic equation in x.	$yx^2 + 2y = 2x^2 + 2x + 3$ $\Rightarrow (y-2)x^2 - 2x + (2y-3) = 0$	M1 A1		
	Uses discriminant to obtain condition for real roots.	For real x $4 - 4(y-2)(2y-3) \geq 0$ $\Rightarrow (2y-5)(y-1) \leq 0$	M1		
		$\Rightarrow 1 \leq y \leq \frac{5}{2}$ (AG)	A1	4	
	Differentiates and equates to zero.	$y' = 0$ $\Rightarrow (x^2+2)(4x+2) - 2x(2x^2+2x+3) = 0$	M1		
	Solves equation.	$\Rightarrow (x-2)(x+1) = 0 \Rightarrow x = -1$ or $x = 2$			
		(Or substitutes $y = 1$ and $\frac{5}{2}$ in equation of C.)			
	States coordinates of turning points.	Turning points are (–1, 1) and $\left(2, 2\frac{1}{2}\right)$	A1A1	3	
	Expresses y in an appropriate form. (May alternatively divide numerator and denominator by x^2.	$y = 2 + \frac{2x-1}{x^2+2}$	M1		
		As $x \to \pm\infty$ $y \to 2$ $\therefore y = 2$	A1	2	
	Finds y-intercept and intersection with $y = 2$.	Shows $\left(0, 1\frac{1}{2}\right)$ and $\left(\frac{1}{2}, 2\right)$	B1		
	Completes graph.	Completely correct graph.	B1	2	**[11]**

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Q30

7	Finds asymptotes to C.	Vertical asymptote $x = 2$.	B1		
		$y = \lambda x + 1 + \frac{2}{x-2} \Rightarrow y = \lambda x + 1$ is oblique asymptote.	M1A1	3	
	Differentiates and equates to 0.	$y' = \lambda - 2(x-2)^{-2} = 0$ for turning points.	M1A1		
		$\lambda = \frac{2}{(x-2)^2} > 0 \Rightarrow$ no turning points if $\lambda < 0$.	A1	3	
		Or $y' = 0 \Rightarrow \lambda x^2 - 4\lambda x + 4\lambda - 2 = 0$	(M1A1)		
		Uses discriminant to show $8\lambda < 0 \Rightarrow$ no T.P.s.	(A1)		
	Sketch of graph. Deduct 1 mark for poor forms at infinity. Deduct 1 mark if intersections with axes not shown.	Axes and asymptotes. LH branch. RH branch. (Indicating intersections with axes at (0,0) and (3,0).)	B1 B1B1	3	**[9]**

Q31

9	States vertical asymptote.	Vertical asymptote is $x = 2$.	B1		
	Divides and states oblique asymptote.	$\Rightarrow$ oblique asymptote is $y = x - 1$.	M1 A1	3	
	Rearranges as quadratic in x.	$xy - 2y = x^2 - 3x + 3 \Rightarrow x^2 - (y+3)x + (3+2y) = 0$	B1		
	Uses discriminant to obtain condition stated.	For real x, $B^2 - 4AC \geq 0$ $\therefore (y+3)^2 - 4(3+2y) \geq 0$ $\Rightarrow ... \Rightarrow (y-3)(y+1)) \geq 0$ $\Rightarrow y \leq -1$ or $y \geq 3$ $\therefore$ no points for $-1 < y < 3$ (AG)	 M1 A1 A1	 4	
	Differentiates, puts = 0 and obtains x-values.	$y' = 1 - (x-2)^{-2} = 0 \Rightarrow x = 1$ or 3 (Or uses $y = -1$ and 3 to obtain x-values.)	 M1		
	States stationary points.	Stationary points are (1,–1) and (3,3)	A1A1	3	
	Sketch.	One mark for each branch correctly placed.	B1B1	2	**[12]**

Q32

6	States vertical asymptote.	Vertical asymptote is $x = 2$.	B1		
	Finds oblique asymptote.	$y = x + 2 + \dfrac{4}{x-2}$ Oblique asymptote is $y = x + 2$.	M1 A1	3	
	Differentiates and equates to zero.	$y' = 1 - \dfrac{4}{(x-2)^2} = 0 \Rightarrow (x-2)^2 = 4$	M1		
	Finds x coordinates.	$x = 0, 4$.	A1		
	States coordinates of turning points.	Turning points are (0,0) and (4,8)	A1	3	
	Deduct at most 1 mark for poor forms at infinity.	Axes and both asymptotes correct. Upper branch correct. Lower branch correct.	B1 B1 B1	3	**[9]**

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SUMMATION OF SERIES

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Q1

2 **(a)** Use standard results from the List of formulae (MF19) to find $\sum_{r=1}^{n}\left(1-r-r^2\right)$ in terms of n, simplifying your answer. [3]

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(b) Show that

$$\frac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)} = \frac{r+1}{\left(r+1\right)^2+1} - \frac{r}{r^2+1}$$

and hence use the method of differences to find $\sum_{r=1}^{n} \frac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)}$. [5]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)}$. [1]

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Q2

1 **(a)** Show that

$$\tan(r+1) - \tan r = \frac{\sin 1}{\cos(r+1)\cos r}. \qquad [2]$$

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Let $u_r = \dfrac{1}{\cos(r+1)\cos r}$.

(b) Use the method of differences to find $\sum_{r=1}^{n} u_r$. [3]

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(c) Explain why the infinite series $u_1 + u_2 + u_3 + \ldots$ does not converge. [1]

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Q3

2 **(a)** Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^{n} r(r+1)(r+2)$ in terms of n, fully factorising your answer. [3]

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(b) Express $\frac{1}{r(r+1)(r+2)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}.$$ [5]

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(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$. [1]

Q4

3 Let $S_n = \sum_{r=1}^{n} \ln \frac{r(r+2)}{(r+1)^2}$.

(a) Using the method of differences, or otherwise, show that $S_n = \ln \frac{n+2}{2(n+1)}$. [4]

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Let $S = \sum_{r=1}^{\infty} \ln \frac{r(r+2)}{(r+1)^2}$.

(b) Find the least value of n such that $S_n - S < 0.01$. [3]

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(c) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha+r)^3+(\beta+r)^3+(\gamma+r)^3\right)=n+\tfrac{1}{4}n(n+1)\left(an^2+bn+c\right),$$

where a, b and c are constants to be determined. [6]

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Q5

2 **(a)** Use standard results from the List of Formulae (MF19) to show that

$$\sum_{r=1}^{n}(7r+1)(7r+8) = an^3 + bn^2 + cn,$$

where a, b and c are constants to be determined. [3]

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(b) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(7r+1)(7r+8)}$ in terms of n. [4]

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(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{(7r+1)(7r+8)}$. [1]

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Q6

3 **(a)** By simplifying $\left(x^n - \sqrt{x^{2n}+1}\right)\left(x^n + \sqrt{x^{2n}+1}\right)$, show that $\dfrac{1}{x^n - \sqrt{x^{2n}+1}} = -x^n - \sqrt{x^{2n}+1}$. [1]

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Let $u_n = x^{n+1} + \sqrt{x^{2n+2}+1} + \dfrac{1}{x^n - \sqrt{x^{2n}+1}}$.

(b) Use the method of differences to find $\sum_{n=1}^{N} u_n$ in terms of N and x. [3]

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(c) Deduce the set of values of x for which the infinite series

$$u_1 + u_2 + u_3 + \ldots$$

is convergent and give the sum to infinity when this exists. [3]

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Q7

4 **(a)** By first expressing $\frac{1}{r^2-1}$ in partial fractions, show that

$$\sum_{r=2}^{n} \frac{1}{r^2-1} = \frac{3}{4} - \frac{an+b}{2n(n+1)},$$

where a and b are integers to be found. [5]

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(b) Deduce the value of $\sum_{r=2}^{\infty} \frac{1}{r^2-1}$. [1]

(c) Find the limit, as $n \to \infty$, of $\sum_{r=n+1}^{2n} \frac{n}{r^2-1}$. [4]

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Q8

3 Let $S_n = 2^2 + 6^2 + 10^2 + \ldots + (4n-2)^2$.

(a) Use standard results from the List of Formulae (MF19) to show that $S_n = \frac{4}{3}n(4n^2 - 1)$. [4]

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(b) Express $\frac{n}{S_n}$ in partial fractions and find $\sum_{n=1}^{N} \frac{n}{S_n}$ in terms of N. [4]

(c) Deduce the value of $\sum_{n=1}^{\infty} \frac{n}{S_n}$. [1]

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Q9

5 Let $S_N = \sum_{r=1}^{N}(5r+1)(5r+6)$ and $T_N = \sum_{r=1}^{N}\frac{1}{(5r+1)(5r+6)}$.

(i) Use standard results from the List of Formulae (MF10) to show that

$$S_N = \tfrac{1}{3}N(25N^2 + 90N + 83).$$ [3]

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(ii) Use the method of differences to express T_N in terms of N. [4]

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(iii) Find $\lim_{N\to\infty}\left(N^{-3}S_N T_N\right)$. [2]

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Q10

2 Let $u_n = \dfrac{4\sin\left(n-\frac{1}{2}\right)\sin\frac{1}{2}}{\cos(2n-1)+\cos 1}$.

(i) Using the formulae for $\cos P \pm \cos Q$ given in the List of Formulae MF10, show that

$$u_n = \frac{1}{\cos n} - \frac{1}{\cos(n-1)}. \qquad [2]$$

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(ii) Use the method of differences to find $\sum_{n=1}^{N} u_n$. [2]

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(iii) Explain why the infinite series $u_1 + u_2 + u_3 + \ldots$ does not converge. [1]

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Q11

4 **(i)** Use the method of differences to show that $\sum_{r=1}^{N} \frac{1}{(3r+1)(3r-2)} = \frac{1}{3} - \frac{1}{3(3N+1)}$. [4]

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(ii) Find the limit, as $N \to \infty$, of $\displaystyle\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r-2)}$. [4]

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Q12

5 Let $S_n = \sum_{r=1}^{n} (-1)^{r-1} r^2$.

(i) Use the standard result for $\sum_{r=1}^{n} r^2$ given in the List of Formulae (MF10) to show that

$$S_{2n} = -n(2n + 1).$$ [4]

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(ii) State the value of $\lim_{n\to\infty} \frac{S_{2n}}{n^2}$ and find $\lim_{n\to\infty} \frac{S_{2n+1}}{n^2}$. [4]

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Q13

2 **(i)** Verify that

$$\frac{n(\mathrm{e}-1)+\mathrm{e}}{n(n+1)\mathrm{e}^{n+1}} = \frac{1}{n\mathrm{e}^{n}} - \frac{1}{(n+1)\mathrm{e}^{n+1}}.$$ [1]

Let $S_N = \sum_{n=1}^{N} \frac{n(\mathrm{e}-1)+\mathrm{e}}{n(n+1)\mathrm{e}^{n+1}}$.

(ii) Express S_N in terms of N and e. [2]

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Let $S = \lim_{N \to \infty} S_N$.

(iii) Find the least value of N such that $(N + 1)(S - S_N) < 10^{-3}$. [3]

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Q14

11 Answer only **one** of the following two alternatives.

EITHER

(i) By considering $(2r + 1)^2 - (2r - 1)^2$, use the method of differences to prove that

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n + 1).$$ [3]

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(ii) By considering $(2r+1)^4 - (2r-1)^4$, use the method of differences and the result given in part **(i)** to prove that

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2. \qquad [5]$$

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The sums S and T are defined as follows:

$$S = 1^3 + 2^3 + 3^3 + 4^3 + \ldots + (2N)^3 + (2N + 1)^3,$$
$$T = 1^3 + 3^3 + 5^3 + 7^3 + \ldots + (2N - 1)^3 + (2N + 1)^3.$$

(iii) Use the result given in part **(ii)** to show that $S = (2N + 1)^2(N + 1)^2$. [1]

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(iv) Hence, or otherwise, find an expression in terms of N for T, factorising your answer as far as possible. [2]

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(v) Deduce the value of $\frac{S}{T}$ as $N \to \infty$. [2]

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Q15

7 Let

$$S_N = \sum_{r=1}^{N}(3r+1)(3r+4) \quad \text{and} \quad T_N = \sum_{r=1}^{N}\frac{1}{(3r+1)(3r+4)}.$$

(i) Use standard results from the List of Formulae (MF10) to show that

$$S_N = N(3N^2 + 12N + 13). \qquad [3]$$

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(ii) Use the method of differences to show that

$$T_N = \frac{1}{12} - \frac{1}{3(3N+4)}. \qquad [3]$$

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(iii) Deduce that $\frac{S_N}{T_N}$ is an integer. [2]

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(iv) Find $\lim_{N\to\infty} \frac{S_N}{N^3 T_N}$. [2]

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Q16

1 It is given that $\sum_{r=1}^{n} u_r = n^2(2n + 3)$, where n is a positive integer.

(i) Find $\sum_{r=n+1}^{2n} u_r$. [2]

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(ii) Find u_r. [3]

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Q17

2 **(i)** Verify that $\dfrac{2r+1}{r(r+1)(r+2)} = \dfrac{1}{2}\left\{\dfrac{(2r+1)(2r+3)}{(r+1)(r+2)} - \dfrac{(2r-1)(2r+1)}{r(r+1)}\right\}$. [2]

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(ii) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{2r+1}{r(r+1)(r+2)} = \frac{1}{2}\left\{\frac{(2n+1)(2n+3)}{(n+1)(n+2)} - \frac{3}{2}\right\}$. [2]

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(iii) Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{2r+1}{r(r+1)(r+2)}$. [2]

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Q18

1 Find $\sum_{r=1}^{n}(4r-3)(4r+1)$, giving your answer in its simplest form. [4]

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Q19

2 Express $\frac{4}{r(r+1)(r+2)}$ in partial fractions and hence find $\sum_{r=1}^{n} \frac{4}{r(r+1)(r+2)}$. [5]

Deduce the value of $\sum_{r=1}^{\infty} \frac{4}{r(r+1)(r+2)}$. [1]

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Q20

1 Verify that $\dfrac{1}{(3r+1)(3r+4)} = \dfrac{1}{3}\left(\dfrac{1}{3r+1} - \dfrac{1}{3r+4}\right)$. [1]

Let S_N denote $\displaystyle\sum_{r=1}^{N} \frac{1}{(3r+1)(3r+4)}$ and let S denote $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(3r+1)(3r+4)}$. Find the least value of N such that $S - S_N < \frac{1}{10\,000}$. [5]

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Q21

1 Use the method of differences to find $\dfrac{1}{(2r)^2 - 1}$. [4]

Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1}$. [1]

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Q22

1 Use the List of Formulae (MF10) to show that $\sum_{r=1}^{13}(3r^2 - 5r + 1)$ and $\sum_{r=0}^{9}(r^3 - 1)$ have the same numerical value. [4]

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4 The sequence $a_1, a_2, a_3, \ldots$ is such that, for all positive integers n,

$$a_n = \frac{n+5}{\sqrt{(n^2-n+1)}} - \frac{n+6}{\sqrt{(n^2+n+1)}}.$$

The sum $\sum_{n=1}^{N} a_n$ is denoted by S_N. Find

(i) the value of S_{30} correct to 3 decimal places, [3]

(ii) the least value of N for which $S_N > 4.9$. [4]

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2 Expand and simplify $(r+1)^4 - r^4$. [1]

Use the method of differences together with the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$ [4]

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Q25

2 Show that the difference between the squares of consecutive integers is an odd integer. [1]

Find the sum to n terms of the series

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \ldots + \frac{2r+1}{r^2(r+1)^2} + \ldots$$

and deduce the sum to infinity of the series. [5]

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1 Given that

$$u_k = \frac{1}{\sqrt{(2k-1)}} - \frac{1}{\sqrt{(2k+1)}},$$

express $\sum_{k=13}^{n} u_k$ in terms of n. [4]

Deduce the value of $\sum_{k=13}^{\infty} u_k$. [1]

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5 Use the method of differences to show that $\sum_{r=1}^{N} \frac{1}{(2r+1)(2r+3)} = \frac{1}{6} - \frac{1}{2(2N+3)}$. [5]

Deduce that $\sum_{r=N+1}^{2N} \frac{1}{(2r+1)(2r+3)} < \frac{1}{8N}$. [4]

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Q28

1 Let $\mathrm{f}(r) = r!(r-1)$. Simplify $\mathrm{f}(r+1) - \mathrm{f}(r)$ and hence find $\sum_{r=n+1}^{2n} r!(r^2+1)$. [5]

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3 It is given that

$$S_n = \sum_{r=1}^{n} u_r = 2n^2 + n.$$

Write down the values of S_1, S_2, S_3, S_4. Express u_r in terms of r, justifying your answer. [4]

Find

$$\sum_{r=n+1}^{2n} u_r.$$ [3]

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1 Express $\dfrac{1}{r(r+1)(r-1)}$ in partial fractions. [1]

Find

$$\sum_{r=2}^{n} \frac{1}{r(r+1)(r-1)}.$$ [4]

State the value of

$$\sum_{r=2}^{\infty} \frac{1}{r(r+1)(r-1)}.$$ [1]

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3 Given that $\mathrm{f}(r) = \dfrac{1}{(r+1)(r+2)}$, show that

$$\mathrm{f}(r-1) - \mathrm{f}(r) = \frac{2}{r(r+1)(r+2)}. \qquad [2]$$

Hence find $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$. [3]

Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$. [1]

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Q32

4 Let $f(r) = r(r+1)(r+2)$. Show that

$$f(r) - f(r-1) = 3r(r+1). \quad [1]$$

Hence show that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$. [2]

Using the standard result for $\sum_{r=1}^{n} r$, deduce that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. [2]

Find the sum of the series

$$1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 + \ldots + 2(n-1)^2 + n^2,$$

where n is odd. [3]

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Q33

1 Show that $\sum_{r=n+1}^{2n} r^2 = \frac{1}{6}n(2n+1)(7n+1)$. [4]

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1 Find the sum of the first n terms of the series

$$\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+\ldots$$

and deduce the sum to infinity. [5]

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SUMMATION OF SERIES
MS

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Q1

Question	Answer	Marks	Guidance
2(a)	$n-\frac{1}{2}n(n+1)-\frac{1}{6}n(n+1)(2n+1)$	**M1A1**	Substitutes correct formulae from MF19. Expanding brackets correctly.
	$\frac{1}{3}n-n^2-\frac{1}{3}n^3$	**A1**	Simplifies by collecting terms.
		3	
2(b)	$\frac{r+1}{(r+1)^2+1}-\frac{r}{r^2+1}=\frac{(r+1)\left(r^2+1\right)-r\left(r^2+2r+2\right)}{\left(r^2+2r+2\right)\left(r^2+1\right)}=\frac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)}$	**M1 A1**	Puts over a common denominator and expands, AG.
	$\sum_{r=1}^{n}\frac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)}=\sum_{r=1}^{n}\left(\frac{r+1}{(r+1)^2+1}-\frac{r}{r^2+1}\right)$ $=\frac{2}{5}-\frac{1}{2}+\frac{3}{10}-\frac{2}{5}+\frac{4}{17}-\frac{3}{10}+\cdots+\frac{n+1}{(n+1)^2+1}-\frac{n}{n^2+1}$	**M1 A1**	Shows at least three complete terms including first and last. Cancellation may be implicit.
	$=-\frac{1}{2}+\frac{n+1}{(n+1)^2+1}$	**A1**	ISW
		5	
2(c)	$-\frac{1}{2}$	**B1 FT**	FT from their answer to part **(b)**.
		1	

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Q2

Question	Answer	Marks	Guidance
1(a)	$\frac{\sin(r+1)}{\cos(r+1)}-\frac{\sin r}{\cos r}=\frac{\sin(r+1)\cos r-\cos(r+1)\sin r}{\cos(r+1)\cos r}=\frac{\sin(r+1-r)}{\cos(r+1)\cos r}$ OR $\frac{\sin(r+1)}{\cos(r+1)}-\frac{\sin r}{\cos r}=\frac{\sin r\cos 1+\cos r\sin 1}{\cos(r+1)}-\frac{\sin r}{\cos r}$ $=\frac{\sin r\cos r\cos 1+\cos^2 r\sin 1-\sin r\cos r\cos 1+\sin^2 r\sin 1}{\cos(r+1)\cos r}$	**M1**	Applies all relevant correct addition formulae from MF19 for the route chosen and combines into a single fraction.
	$=\frac{\sin 1}{\cos(r+1)\cos r}$ $\frac{\tan r+\tan 1}{1-\tan r\tan 1}-\tan r=\frac{\tan r+\tan 1-\tan r+\tan^2 r\tan 1}{1-\tan r\tan 1}$ $=\frac{\tan 1\sec^2 r}{1-\tan r\tan 1}=\frac{\tan 1}{\cos r(\cos r-\sin r\tan 1)}=\frac{\sin 1}{\cos r(\cos r\cos 1-\sin r\sin 1)}$ $=\frac{\sin 1}{\cos r\cos(r+1)}$	**A1**	SC B2 for this route completely correct to AG.
		2	
1(b)	$[\sin 1]\sum_{r=1}^{n}u_r=\tan 2-\tan 1+\tan 3-\tan 2+\ldots+\tan(n+1)-\tan n$	**M1 A1**	Shows enough terms for cancellation to be clear. There must be three complete terms including first and last. Cancelling may be implied.
	$\sum_{r=1}^{n}u_r=\frac{\tan(n+1)-\tan 1}{\sin 1}$	**A1**	OE and ISW.
		3	
1(c)	$\tan(n+1)$ oscillates as $n\to\infty$ so $u_1+u_2+u_3+\ldots$ does not converge.	**B1**	States 'oscillates' or refers to diverging values of $\tan(n+1)$, or states that $\tan(n+1)$ does not tend to a limit.
		1	

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Q3

Question	Answer	Marks	Guidance
2(a)	$\sum_{r=1}^{n}\left(r^3+3r^2+2r\right)=\frac{1}{4}n^2(n+1)^2+\frac{1}{2}n(n+1)(2n+1)+n(n+1)$	**M1 A1**	Expands and substitutes formulae. At least two formulae used for M1.
	$=\frac{1}{4}n(n+1)\left(n^2+n+4n+2+4\right)=\frac{1}{4}n(n+1)(n+2)(n+3)$	**A1**	
		3	
2(b)	$\frac{1}{r(r+1)(r+2)}=\frac{1}{2r}-\frac{1}{r+1}+\frac{1}{2(r+2)}$	**M1 A1**	Finds partial fractions.
	$\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\ \frac{1}{2}f(1)-f(2)+\frac{1}{2}f(3)$ $+\frac{1}{2}f(2)-f(3)+\frac{1}{2}f(4)$ $+\frac{1}{2}f(3)-f(4)+\frac{1}{2}f(5)$ $\vdots$ $+\frac{1}{2}f(n-1)-f(n)+\frac{1}{2}f(n+1)$ $+\frac{1}{2}f(n)-f(n+1)+\frac{1}{2}f(n+2)$	**M1 A1**	Shows enough terms for cancelation to be clear. Where $f(x)=\frac{1}{r}$
	$=\frac{1}{4}-\frac{1}{2(n+1)}+\frac{1}{2(n+2)}$	**A1**	ISW
		5	
2(c)	$\frac{1}{4}$	**B1 FT**	FT from *their* answer to part **(b)**.
		1	

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Q4

Question	Answer	Marks	Guidance
3(a)	$\sum_{r=1}^{n}\left(\ln r - 2\ln(r+1) + \ln(r+2)\right)$	**B1**	Separates logarithms into correct form using a difference. Or as logarithm of product.
	$\ln 1 - 2\ln 2 + \ln 3$ $\ln 2 - 2\ln 3 + \ln 4$ $\ln 3 - 2\ln 4 + \ln 5$ $\vdots$ $\ln(n-1) - 2\ln n + \ln(n+1)$ $\ln n - 2\ln(n+1) + \ln(n+2)$	**M1 A1**	Shows enough terms to make cancellation clear.
	$[\ln 1] - \ln 2 - \ln(n+1) + \ln(n+2) = \ln\dfrac{n+2}{2(n+1)}.$	**A1**	AG
		4	
3(b)	$S = -\ln 2$	**B1**	States sum to infinity. AEF
	$S_n - S = \ln\left(\dfrac{n+2}{n+1}\right) < 0.01$ leading to $n+2 < \mathrm{e}^{0.01}(n+1)$	**M1**	Forms inequality
	Least value of n is 99	**A1**	CAO
		3	

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Q5

Question	Answer	Marks	Guidance
2(a)	$\sum_{r=1}^{n}(7r+1)(7r+8)=\sum_{r=1}^{n}49r^2+63r+8$	**M1**	Expands.
	$=49\left(\frac{1}{6}n(n+1)(2n+1)\right)+63\left(\frac{1}{2}n(n+1)\right)+8n$	**M1**	Substitutes formulae for $\sum r^2$ and $\sum r$.
	$=\frac{49}{3}n^3+56n^2+\frac{143}{3}n$	**A1**	
		3	
2(b)	$\frac{1}{(7r+1)(7r+8)}=\frac{1}{7}\left(\frac{1}{7r+1}-\frac{1}{7r+8}\right)$	**M1 A1**	Finds partial fractions.
	$\sum_{r=1}^{n}\frac{1}{(7r+1)(7r+8)}=\frac{1}{7}\left(\frac{1}{8}-\frac{1}{15}+\frac{1}{15}-\frac{1}{22}+\ldots+\frac{1}{7n+1}-\frac{1}{7n+8}\right)$	**M1**	Writes at least three correct terms, including first and last.
	$=\frac{1}{7}\left(\frac{1}{8}-\frac{1}{7n+8}\right)$	**A1**	
		4	
2(c)	$\frac{1}{56}$	**B1 FT**	
		1	

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Q6

Question	Answer	Marks	Guidance
3(a)	$\left(x^n-\sqrt{\left(x^{2n}+1\right)}\right)\left(x^n+\sqrt{\left(x^{2n}+1\right)}\right)=x^{2n}-\left(x^{2n}+1\right)=-1$	**B1**	Uses the difference of two squares and obtains the given answer. AG
		1	
3(b)	$u_n=x^{n+1}+\sqrt{\left(x^{2n+2}+1\right)}-x^n-\sqrt{\left(x^{2n}+1\right)}$	**B1**	Uses the identity given in part **(a)**.
	$\sum_{n=1}^{N}u_n=x^2+\sqrt{\left(x^4+1\right)}-x-\sqrt{\left(x^2+1\right)}+x^3+\sqrt{\left(x^6+1\right)}-x^2-\sqrt{\left(x^4+1\right)}$ $+\ldots+x^{N+1}+\sqrt{\left(x^{2N+2}+1\right)}-x^N-\sqrt{\left(x^{2N}+1\right)}$	**M1**	Writes at least three complete correct terms, including the first and last.
	$=x^{N+1}+\sqrt{\left(x^{2N+2}+1\right)}-x-\sqrt{\left(x^2+1\right)}$	**A1**	(Don't allow in terms of n.)
		3	
3(c)	$-1<x<1$	**B1**	
	$x^{N+1}\to 0$ as $n\to\infty\Rightarrow u_1+u_2+u_3+\ldots=1-x-\sqrt{\left(x^2+1\right)}$	**M1 A1**	Finds sum to infinity.
		3	

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Q7

Question	Answer	Marks
4(a)	$\frac{1}{r^2-1}=\frac{1}{(r-1)(r+1)}=\frac{1}{2}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$	**M1 A1**
	$\sum_{r=2}^{n}\frac{1}{r^2-1}=\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+\cdots+\frac{1}{n-2}-\frac{1}{n}+\frac{1}{n-1}-\frac{1}{n+1}\right)$	**M1**
	$=\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right)$	**A1**
	$=\frac{3}{4}-\frac{2n+1}{2n(n+1)}$	**A1**
		5
4(b)	$\frac{3}{4}$	**B1**
		1
4(c)	$\sum_{r=n+1}^{2n}\frac{n}{r^2-1}=\sum_{r=2}^{2n}\frac{n}{r^2-1}-\sum_{r=2}^{n}\frac{n}{r^2-1}$	**M1**
	$=n\left(\frac{3}{4}-\frac{4n+1}{4n(2n+1)}\right)-n\left(\frac{3}{4}-\frac{2n+1}{2n(n+1)}\right)=\frac{n(2n+1)}{2n(n+1)}-\frac{n(4n+1)}{4n(2n+1)}$	**M1A1**
	$\to\frac{1}{2}$ as $n\to\infty$	**A1**
		4

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Q8

Question	Answer	Marks
3(a)	$S_n = \sum_{r=1}^{n}(4r-2)^2 = 16\sum_{r=1}^{n}r^2 - 16\sum_{r=1}^{n}r + 4n$	**M1 A1**
	$= \frac{8}{3}n(n+1)(2n+1) - 8n(n+1) + 4n$	**M1**
	$\frac{4}{3}n(4n^2+6n+2-6n-6+3) = \frac{4}{3}n\left(4n^2-1\right)$ **AG**	**A1**
		4
3(b)	$\frac{n}{S_n} = \frac{3}{4(2n-1)(2n+1)} = \frac{3}{8}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$	**M1 A1**
	$\sum_{n=1}^{N}\frac{n}{S_n} = \frac{3}{8}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} \ldots + \frac{1}{2N-1} - \frac{1}{2N+1}\right)$	**M1**
	$= \frac{3}{8}\left(1 - \frac{1}{2N+1}\right)$	**A1**
		4
3(c)	$\frac{3}{8}$	**B1**
		1

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Q9

Question	Answer	Marks	Guidance
5(i)	$\sum_{r=1}^{N}(5r+1)(5r+6)=25\sum_{r=1}^{N}r^2+35\sum_{r=1}^{N}r+6N$	**M1**	Expands.
	$25\left(\frac{1}{6}N(N+1)(2N+1)\right)+35\left(\frac{1}{2}N(N+1)\right)+6N$	**M1**	Substitutes formulae for $\sum r$ and $\sum r^2$.
	$=N\left(\frac{25}{6}(2N^2+3N+1)+\frac{35}{2}N+\frac{35}{2}+6\right)=\frac{1}{3}N(25N^2+90N+83)$	**A1**	Simplifies to the given answer (AG).
		3	
5(ii)	$\frac{1}{(5r+1)(5r+6)}=\frac{1}{5}\left(\frac{1}{5r+1}-\frac{1}{5r+6}\right)$	**M1 A1**	Finds partial fractions.
	$T_N=\frac{1}{5}\left(\frac{1}{6}-\frac{1}{11}+\frac{1}{11}-\frac{1}{16}+\cdots+\frac{1}{5N+1}-\frac{1}{5N+6}\right)$	**M1**	Expresses terms as differences.
	$\frac{1}{5}\left(\frac{1}{6}-\frac{1}{5N+6}\right)=\frac{1}{30}-\frac{1}{5(5N+6)}$	**A1**	At least 3 terms including last.
		4	
5(iii)	$\frac{S_N}{N^3}T_N\to\frac{25}{3}\times\frac{1}{30}=\frac{5}{18}$	**M1 A1**	Divides S_N by N^3 and takes limits as $N\to\infty$
		2	

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Q10

Question	Answer	Marks	Guidance
2(i)	$\dfrac{4\sin\left(n-\frac{1}{2}\right)\sin\frac{1}{2}}{\cos(2n-1)+\cos 1}=\dfrac{2\left(\cos(n-1)-\cos n\right)}{2\cos n\cos(n-1)}$	**M1**	Uses formulae for $\cos P \pm \cos Q$.
	$\dfrac{1}{\cos n}-\dfrac{1}{\cos(n-1)}$	**A1**	AG
		2	
2(ii)	$\displaystyle\sum_{n=1}^{N}\frac{4\sin\left(n-\frac{1}{2}\right)\sin\frac{1}{2}}{\cos(2n-1)+\cos 1}=\sum_{n=1}^{N}\frac{1}{\cos n}-\frac{1}{\cos(n-1)}$ $=\dfrac{1}{\cos 1}-\dfrac{1}{1}+\dfrac{1}{\cos 2}-\dfrac{1}{\cos 1}+\ldots+\dfrac{1}{\cos N}-\dfrac{1}{\cos(N-1)}$	**M1**	Applies (i), shows enough terms and cancelation.
	$-1+\dfrac{1}{\cos N}$	**A1**	
		2	
2(iii)	$\cos N$ oscillates as $N\to\infty$ so $u_1+u_2+u_3+\ldots$ does not converge.	**B1**	States "oscillates" or refers to diverging values of $\cos N$.
		1	

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Q11

Question	Answer	Marks	Guidance
4(i)	$\frac{1}{(3r+1)(3r-2)} = \frac{1}{3}\left(\frac{1}{3r-2} - \frac{1}{3r+1}\right)$	**M1 A1**	Finds partial fractions.
	$\sum_{r=1}^{N} \frac{1}{(3r+1)(3r-2)} = \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4} + \frac{1}{4} - \frac{1}{7} + \ldots \frac{1}{3N-2} - \frac{1}{3N+1}\right)$	**M1**	At least 3 term including final term.
	$= \frac{1}{3}\left(1 - \frac{1}{3N+1}\right)$	**A1**	AG
		4	
4(ii)	$\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r-2)} = \sum_{r=1}^{N^2} \frac{N}{(3r+1)(3r-2)} - \sum_{r=1}^{N} \frac{N}{(3r+1)(3r-2)}$	**M1**	Uses $\sum_{r=N+1}^{N^2} = \sum_{r=1}^{N^2} - \sum_{r=1}^{N}$
	$= \frac{N}{3} - \frac{N}{3\left(3N^2+1\right)} - \left(\frac{N}{3} - \frac{N}{3(3N+1)}\right)$	**M1**	Applies **(i)**
	$= \frac{N}{3(3N+1)} - \frac{N}{3\left(3N^2+1\right)} = \frac{N^3 - N^2}{(3N+1)\left(3N^2+1\right)}$	**A1**	Allow simplification to common denominator.
	$\to \frac{1}{9}$ as $N \to \infty$	**B1**	
		4	

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Q12

Question	Answer	Marks	Guidance
5(i)	$_{2n} = 1^2 - 2^2 + 3^2 - 4^2 \ldots..$	**M1**	Uses correct difference.
	so $S_{2n} = \sum_{r=1}^{2n} r^2 - 2\sum_{r=1}^{n} (2r)^2 = \sum_{r=1}^{2n} r^2 - 8\sum_{r=1}^{n} r^2$	**A1**	Alt method: Use $\sum_1^n (2r-1)^2 - \sum_1^n (2r)^2$ =A1
	Thus $S_{2n} = \frac{1}{6}(2n)(2n+1)(4n+1) - \frac{8}{6}n(n+1)(2n+1)$	**M1**	$\sum_1^n 4r^2 - 4\sum_1^n (r) + n - 4\sum_1^n (r)^2$ M1
	Factorising, $S_{2n} = \frac{1}{3}n(2n+1)(4n+1-4n-4) = -n(2n+1)$	**A1**	$= -n(2n+1)$ A1 AG
		4	
5(ii)	$\lim_{n\to\infty} \frac{S_{2n}}{n^2} = -2$	**B1**	
	$S_{2n+1} = S_{2n} + (-1)^{2n}(2n+1)^2$	**M1**	
	So, $S_{2n+1} = -n(2n+1) + (2n+1)^2 = (2n+1)(n+1)$	**M1**	Uses the result given in **(i)** or using $\lim_{n\to\infty} \frac{S_{2n}}{n^2}$ and correct sign
	Thus $\lim_{n\to\infty} \frac{S_{2n+1}}{n^2} = 2$	**A1**	Alt: Find limit from previous line directly
		4	

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Q13

Question	Answer	Marks	Guidance
2(i)	$\frac{1}{ne^n}-\frac{1}{(n+1)e^{n+1}}=\frac{(n+1)e-n}{n(n+1)e^{n+1}}=\frac{n(e-1)+e}{n(n\ \ 1)e}$	**B1**	Verifies result (AG).
2(ii)	$S_N=\sum_{n=1}^{N}\left(\frac{1}{ne^n}-\frac{1}{(n+1)e^{n+1}}\right)=(\frac{1}{e}-\frac{1}{2e^{\ 2}}+\frac{1}{2e^{\ 2}}-\frac{1}{3e^{\ 3}}+$ ..... $\frac{1}{Ne^{\ N}}-\frac{1}{(N+1)e^{N+1}}$) SOI =	**M1**	Uses difference method to sum.
	$\frac{1}{e}-\frac{1}{(N+1)e^{N+1}}.$	**A1**	
2(iii)	$S=\frac{1}{e}$	**B1**	Finds S
	$(N+1)(S-S_N)$ $<10^{-3}\Rightarrow\frac{1}{e^{N+1}}<10^{-3}$	**M1**	Attempts to find difference between sum and sum to infinity.
	$\Rightarrow e^{N+1}>10^3$ $\Rightarrow$ least such N is 6.	**A1**	
		6	

Q14

Question	Answer	Marks	Guidance
11E(i)	$(2r+1)^2-(2r-1)\ =8r$	**B1**	
	$\Rightarrow\ 8\sum_{r=1}^{n}r=-1^2+(2n+1)^2$	**M1**	Sums both sides and uses method of differences.
	$\Rightarrow\ \sum_{r=1}^{n}r=\frac{1}{2}n(n+1)$	**A1**	AG.
		3	

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11E(ii)	$(2r+1)^4-(2r-1)\ =\left((2r+1)^2+(2r-1)^2\right)\left((2r+1)^2-(2r-1)^2\right)$	**M1**	Uses difference of squares or expands.
	$=\left(8r^2+2\right)(8r)=16\left(4r^3+r\right)$	**A1**	
	$\Rightarrow 4\sum_{r=1}^{n}r^3+\sum_{r=1}^{n}r=-\left(1-\frac{1}{2}\right)^4+\left(n+\frac{1}{2}\right)^4$	**M1**	Sums both sides and uses method of differences.
	$4\sum_{r=1}^{n}r^3=\left(n+\frac{1}{2}\right)^4-\left(\frac{1}{2}\right)^4-\frac{1}{2}n(n+1)$	**M1**	Uses formula for $\sum r$
	$=\left(\left(n+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right)\left(\left(n+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2\right)-\frac{1}{2}n(n+1)=n^2(n+1)^2$	**A1**	AG
		5	
11E(iii)	$S=\frac{1}{4}(2N+1)^2(2N+2)^2=(2N+1)^2(N+1)^2$	**B1**	Uses formula for $\sum r^3$. AG.
		1	
11E(iv)	$T=S-\sum_{r=1}^{N}(2r)^3=(2N+1)^2(N+1)^2-2N^2(N+1)^2$	**M1**	Eliminates even terms from S.
	$(N+1)^2\left(2N^2+4N+1\right)$	**A1**	Accept $(\mathrm{N}+1)^2\left(2N(N+2)+1\right)$.
		2	
11E(v)	$\frac{S}{T}=\frac{(2N+1)^2}{2N^2+4N+1}=\frac{4N^2+4N+1}{2N^2+4N+1}$	**M1**	Writes fraction as quadratic in N divided by quadratic in N .
	Converges to 2 as $N\to\infty$.	**A1**	
		2	

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Q15

Question	Answer	Marks	Guidance
7(i)	$\sum_{r=1}^{N}(3r+1)(3r+4)=9\sum_{r=1}^{N}r^2+15\sum_{r=1}^{N}r+4N$	**M1**	Expands
	$9\left(\frac{1}{6}N(N+1)(2N+1)\right)+15\left(\frac{1}{2}N(N+1)\right)+4N$	**M1**	Substitutes formulae for $\sum r$ and $\sum r^2$.
	$=N\left(\frac{9}{6}(2N^2+3N+1)+\frac{15}{2}N+\frac{15}{2}+4\right)$ $=N(3N^2+12N+13)$	**A1**	Shows simplification to the given answer (AG).
		3	
7(ii)	$\frac{1}{(3r+1)(3r+4)}=\frac{1}{3}\left(\frac{1}{3r+1}-\frac{1}{3r+4}\right)$	**B1**	Finds partial fractions.
	$T_N=\frac{1}{3}\left(\frac{1}{4}-\frac{1}{7}+\frac{1}{7}-\frac{1}{10}+\cdots+\frac{1}{3(N-1)+1}-\frac{1}{3N+4}\right)$	**M1**	Expresses terms as differences.
	$\frac{1}{3}\left(\frac{1}{4}-\frac{1}{3N+4}\right)=\frac{1}{12}-\frac{1}{3(3N+4)}$	**A1**	Cancels to given answer (AG).
		3	
7(iii)	$T_N=\frac{N}{4(3N+4)}\Rightarrow\ \frac{S_N}{T_N}=4(3N+4)(3N^2+12N+13)$	**M1**	Writes $\frac{S_N}{T_N}$ as a polynomial
	So $\frac{S_N}{T_N}$ is an integer because all terms are integers	**A1**	Justifies expression being integer
		2	

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7(iv)	$\frac{S_N}{N^3 T_N} = \frac{4(3N+4)(3N^2+16N+9)}{N^3}$	**M1**	Divides expression in **(iii)** by N^3 and takes limit
	$\to 4(3)(3) = 36$	**A1**	
		2	

Q16

Question	Answer	Marks	Guidance
1(i)	$\sum_{n+1}^{2n} u_r = (2n)^2(4n+3) - n^2(2n+3)$	**M1**	Method mark for using $S_{2n} - S_n$
	$= 14n^3 + 9n^2$	**A1**	
	Total:	**2**	
1(ii)	$u_r = r^2(2r+3) - (r-1)^2(2r+1)$	**M1A1**	Method mark for using $S_r - S_{r-1}$ OE
	$= 6r^2 - 1$	**A1**	SR: CAO **B1** without wrong working
	Total:	**3**	

Q17

Question	Answer	Marks	Guidance
2(i)	$\text{RHS} = \frac{1}{2}\left\{\frac{4r^3+8r^2+3r-(4r^2-1)(r+2)}{r(r+1)(r+2)}\right\}$	**M1**	
	$= \frac{1}{2}\left\{\frac{4r^3+8r^2+3r-(4r^3+8r^2-r-2)}{r(r+1)(r+2)}\right\} = \frac{1}{2}\left\{\frac{(4r+2)}{r(r+1)(r+2)}\right\} = \frac{(2r+1)}{r(r+1)(r+2)}$	**A1**	AG
	Total:	**2**	

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2(ii)	to n terms is: $\frac{1}{2}\left\{\left[\frac{3.5}{2.3}-\frac{1.3}{1.2}\right]+\left[\frac{5.7}{3.4}-\frac{3.5}{2.3}\right]+\ldots+\left[\frac{(2n-1)(2n+1)}{n(n+1)}-\frac{(2n-3)(2n-1)}{n(n-1)}\right]+\left[\frac{(2n+1)(2n+3)}{(n+1)(n+2)}-\frac{(2n-1)(2n+1)}{n(n+1)}\right]\right\}$	**M1**	
	$=\frac{1}{2}\left\{\frac{(2n+1)(2n+3)}{(n+1)(n+2)}-\frac{3}{2}\right\}$	**A1**	AG
2(iii)	$S_\infty=\frac{1}{2}\times 4-\frac{3}{4}=1\frac{1}{4}$.	**M1A1**	
	Total:	**4**	

Q18

Question	Answer	Marks	Guidance
1	$\sum_{r=1}^{n}u_r=16\sum_{r=1}^{n}r^2-8\sum_{r=1}^{n}r-3n$	**M1A1**	**M1** for split into 3 parts
	$=16\frac{n(n+1)(2n+1)}{6}-8\frac{n(n+1)}{2}-3n$	**M1**	For using formulae correctly in their expression
	$=\ldots=\frac{n}{3}\left(16n^2+12n-13\right)$ (3 terms)	**A1**	OE
		4	

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Q19

2	$\frac{2}{r}-\frac{4}{r+1}+\frac{2}{r+2}$ (Award **B2** if written down by cover up rule.)	**M1A1**
	$\left(2-2+\frac{2}{3}\right)+\left(1-\frac{4}{3}+\frac{1}{2}\right)+\ldots+\left(\frac{2}{n-1}-\frac{4}{n}+\frac{2}{n+1}\right)+\left(\frac{2}{n}-\frac{4}{n+1}+\frac{2}{n+2}\right)$	**M1A1**
	$=1-\frac{2}{n+1}+\frac{2}{n+2}$ (AEF)	**A1** [5]
	Sum to infinity = 1	**B1** [1]

Q20

1	$\frac{1}{3}\left(\frac{1}{3r+1}-\frac{1}{3r+4}\right)=\frac{1}{3}\left(\frac{(3r+4)-(3r+1)}{(3r+1)(3r+4)}\right)=\frac{1}{(3r+1)(3r+4)}$ **AG**	**B1** [1]
	$S_N=\frac{1}{3}\left[\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\ldots+\left(\left(\frac{1}{3N+1}\right)-\frac{1}{3N+4}\right)\right]=\frac{1}{3}\left(\frac{1}{4}-\frac{1}{3N+4}\right)$	**M1 A1**
	$\Rightarrow S=\frac{1}{12}$	**A1**
	$\Rightarrow S-S_N=\frac{1}{3(3N+4)}<\frac{1}{10000}$	**M1**
	$\Rightarrow 3N+4>\frac{10000}{3}\Rightarrow N>1109\frac{7}{9}$. Thus least N is 1110.	**A1** [5]

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Q21

1	$\frac{1}{(2r-1)(2r+1)} = \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$ $\sum_{r=1}^{n} \frac{1}{(2r)^2-1} = \frac{1}{2}\left(\left[1-\frac{1}{3}\right]+\left[\frac{1}{3}-\frac{1}{5}\right]+\ldots+\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]\right) = \frac{1}{2}\left(1-\frac{1}{2n+1}\right)$ (OE)	**M1A1** **M1A1**	[4]
	$\frac{1}{2}\left(1-\frac{1}{2n+1}\right) = \frac{n}{2n+1} \Rightarrow \sum_{r=1}^{\infty} \frac{1}{(2r)^2-1} = \frac{1}{2}$	**B1**√	[1]

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Q22

1	$3\times\frac{13\times14\times27}{6}-5\times\frac{13\times14}{2}+13=2015$ $\left[\frac{9\times10}{2}\right]^2-10=2015$ (Award M1 for subtracting 9 or 10 here.)	M1A1 M1A1 (4) **Total: 4**

Q23

4	**(i)**	$\left(\frac{6}{\sqrt{1}}-\frac{7}{\sqrt{3}}\right)+\left(\frac{7}{\sqrt{3}}-\frac{8}{\sqrt{7}}\right)+\ldots+\left(\frac{35}{\sqrt{871}}-\frac{36}{\sqrt{931}}\right)=6-\frac{36}{\sqrt{931}}=4.820$	**M1A1** **A1** [3]
	(ii)	$6-\frac{n+6}{\sqrt{n^2+n+1}}>4.9\Rightarrow 0.21n^2-10.79n-34.79(>0)$ $\Rightarrow n>54.42\ldots$ so 55 terms required.	**M1A1** **dM1A1** [4] **Total 7**

Q24

2	$(r+1)^4-r^4=4r^3+6r^2+4r+1$	B1 [1]
	$(n+1)^4-1^4=4\Sigma_{r=1}^{n}r^3+6\Sigma_{r=1}^{n}r^2+4\Sigma_{r=1}^{n}r+n$	M1
	$n^4+4n^3+6n^2+4n=4\Sigma_{r=1}^{n}r^3+n(2n^2+3n+1)+2n^2+2n+n$	A1A1
	$\Rightarrow\ldots\Rightarrow\Sigma_{r=1}^{n}r^3=\frac{1}{4}n^2(n+1)^2$. (AG)	A1 [4]

Q25

2	$(n+1)^2-n^2=n^2+2n+1-n^2=2n+1\Rightarrow$ odd.	B1 [1]
	$\frac{3}{1^2.2^2}+\frac{5}{2^2.3^2}+\frac{7}{3^2.4^2}+\ldots\frac{2n+1}{n^2(n+1)^2}=\frac{2^2-1^2}{1^2.2^2}+\frac{3^2-2^2}{2^2.3^2}+\frac{4^2-3^2}{3^2.4^2}+\ldots\frac{(n+1)^2-n^2}{n^2(n+1)^2}$	M1A1
	$=1-\frac{1}{2^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots+\frac{1}{n^2}-\frac{1}{(n+1)^2}$	M1
	$=1-\frac{1}{(n+1)^2}$	A1
	Sum to infinity =1.	A1√ [5]

Q26

1	$\left(\frac{1}{\sqrt{25}}-\frac{1}{\sqrt{27}}\right)+\left(\frac{1}{\sqrt{27}}-\frac{1}{\sqrt{29}}\right)+\ldots+\left(\frac{1}{\sqrt{2n-1}}-\frac{1}{\sqrt{2n+1}}\right)$	M1A1
	$\sum_{r=13}^{n}u_k=\frac{1}{5}-\frac{1}{\sqrt{2n+1}}$	M1A1 (4)
	$\sum_{r=13}^{\infty}u_k=\frac{1}{5}$	B1√ (1) **[5]**

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Q27

5	Finds partial fractions.	$\frac{1}{(2r+1)(2r+3)} = \frac{1}{2}\left\{\frac{1}{2r+1} - \frac{1}{2r+3}\right\}$ $\sum_{r=1}^{N} \frac{1}{(2r+1)(2r+3)}$	M1A1		
	Expresses terms as differences.	$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \ldots + \frac{1}{2}\left(\frac{1}{2N+1} - \frac{1}{2N+3}\right)$	M1A1		
	Shows cancellation.	$= \frac{1}{6} - \frac{1}{2(2N+3)}$ (AG)	A1		
	Uses $\sum_{N+1}^{2N} = \sum_{1}^{2N} - \sum_{1}^{N}$.	$\sum_{N+1}^{2N} = \left(\frac{1}{6} - \frac{1}{2(4N+3)}\right) - \left(\frac{1}{6} - \frac{1}{2(2N+3)}\right)$	M1		
	Applies result	$= \frac{1}{2}\left(\frac{1}{2N+3} - \frac{1}{4N+3}\right)$	A1		
	and simplifies.	$= \frac{N}{(2N+3)(4N+3)}$	M1		
	Deduces inequality.	$< \frac{N}{2N.4N} = \frac{1}{8N}$ (AG)	A1		**[9]**

Q28

1	Simplifies.	$f(r+1) - f(r) = r(r+1)! - (r-1)r!$ $= r!(r^2 + r - r + 1) = r!(r^2+1)$	M1 A1		
	Uses difference method.	$\sum_{1}^{n} = f(2) - f(1) + f(3) - f(2) + \ldots f(n+1) - f(n)$ $= n(n+1)! - 0 = n(n+1)!$	M1 A1		
	Obtains result.	$\therefore \sum_{n+1}^{2n} = 2n(2n+1)! - n(n+1)!$	A1	5	**[5]**
		(Or directly using $\sum_{n+1}^{2n} = f(2n+1) - f(n+1)$ from the method of differences.)			

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Q29

3	Writes first four sums.	$S_1 \ldots S_4 \sim 3, 10, 21, 36$	B1		
	Deduces first four terms, conjectures and justifies result.	$u_1 \ldots u_4 \sim 3, 7, 11, 15 \Rightarrow u_r = 4r - 1$ since $S_n = \frac{n}{2}\{6 + 4(n-1)\} = 2n^2 + n$ as given.	B1B1 B1		
		Or $u_r = S_r - S_{r-1} = 2r^2 + r - 2(r-1)^2 - (r-1)$ $= 4r - 1$	B1B1 B1	4	
	Obtains required sum.	$\sum_{n+1}^{2n}(4r-1) = 4.\frac{2n(2n+1)}{2} - 2n - \left(4.\frac{n(n+1)}{2} - n\right)$	M1A1		
		$= 8n^2 + 2n - (2n^2 + n) = 6n^2 + n$	A1	3	
		Or Sum of AP $= \frac{n}{2}(4n + 3 + 8n - 1) = 6n^2 + n$			**[7]**

Q30

1	Finds partial fractions.	$\frac{1}{r(r-1)(r+1)} = \frac{1}{2(r-1)} - \frac{1}{r} + \frac{1}{2(r+1)}$	B1	**(1)**	
	Expresses each term in fractions	$\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right) \ldots \left(\frac{1}{2(n-1)} - \frac{1}{n} + \frac{1}{2(n+1)}\right)$	M1A1		
	Cancels terms and sums	$= \frac{1}{4} - \frac{1}{2n} + \frac{1}{2(n+1)}$ (OE)	M1A1	**(4)**	
	Find sums to infinity	$S\infty = \frac{1}{4}$	B1	**(1)**	**[6]**

Q31

3	Proves initial result.	$f(r-1) - f(r) = \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$	M1		
		$= \frac{r+2-r}{r(r+1)(r+2)} = \frac{2}{r(r+1)(r+2)}$ (AG)	A1	2	
	Sets up method of differences.	$\sum_1^n \frac{1}{r(r+1)(r+2)} = \frac{1}{2}\left\{\frac{1}{1\times 2} - \frac{1}{2\times 3}\right\} \ldots\ldots$	M1A1		
		$+ \frac{1}{2}\left\{\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right\}$			
	Shows cancellation to get result.	$= \frac{1}{4} - \frac{1}{2}\left\{\frac{1}{(n+1)(n+2)}\right\}$ (OE)	A1	3	
	States sum to infinity.	$\therefore \sum_1^\infty \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$	A1√	1	**[6]**
	'Non hence' method for last two parts	$\frac{1}{r(r+1)(r-2)} = \frac{1}{2r} - \frac{1}{(r+1)} + \frac{1}{2(r+2)}$	(M1)		
	i.e. penalty of 1 mar	$\Rightarrow \cdots \Rightarrow$			
		$\frac{1}{2} - \frac{1}{2} + \frac{1}{4} \ldots + \frac{1}{2(n+1)} - \frac{1}{(n+1)} + \frac{1}{2(n+2)}$	(A1)		
		$= \frac{1}{4} - \frac{1}{2}\left\{\frac{1}{(n+1)(n+2)}\right\}$ (OE)	(A1)	(3)	
		$\therefore \sum_1^\infty \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$	(A1√)	(1)	

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Q32

4	Verifies given result.	$r(r+1)(r+2)-(r-1)r(r+1)=r(r+1)(r+2-r+1)$ $=3r(r+1)$ (AG)	B1	1	
	Uses method of differences to sum first series.	$\sum_{r=1}^{n} r(r+1) =$ $\frac{1}{3}\{[f(n)-f(n-1)]+[f(n-1)-f(n-2)]+...+[f(1)-f(0)]\}$	M1		
		$=\frac{1}{3}n(n+1)(n+2)$ (AG) (Award B1 if 'not hence'.)	A1	2	
	Subtracts $\sum_{r=1}^{n} r$ to obtain sum of second series.	$\sum_{r=1}^{n} r^2=\sum_{r=1}^{n} r(r+1)-\sum_{r=1}^{n} r=\frac{n(n+1)(n+2)}{3}-\frac{n(n+1)}{2}$	M1		
		$=\frac{1}{6}n(n+1)(2n+4-3)=\frac{1}{6}n(n+1)(2n+1)$ (AG)	A1	2	
	Splits series into two series.	$(1^2+2^2+...+n^2)+(2^2+4^2+...+(n-1)^2)=$	M1		
	Applies sum of squares formula to obtain result.	$\frac{n(n+1)(2n+1)}{6}+\frac{4\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)n}{6}=\ldots=\frac{1}{2}n^2(n+1)$	M1A1	3	**[8]**

Q33

1	Use of:	$\sum_{n+1}^{2n} = \sum_{1}^{2n} - \sum_{1}^{n}$	M1		
	Use of:	$\sum_{1}^{n} r^2=\frac{n(n+1)(2n+1)}{6}$	M1		
	Obtains result.	$\frac{2n(2n+1)(4n+1)}{6}-\frac{n(n+1)(2n+1)}{6}$	A1		
		$=\frac{1}{6}n(2n+1)(8n+2-n-1)=\frac{1}{6}n(2n+1)(7n+1)$ (AG)	A1	4	**[4]**

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Q34

1	Finds partial fractions.	$\frac{1}{r(r+2)}=\frac{1}{2}\left\{\frac{1}{r}-\frac{1}{r+2}\right\}$ $\sum_{r=1}^{n}\frac{1}{r(r+2)} =$	M1A1		
	Use method of differences.	$\frac{1}{2}\left\{\left[\frac{1}{n}-\frac{1}{n+2}\right]+\left[\frac{1}{n-1}-\frac{1}{n+1}\right]+...+\left[\frac{1}{2}-\frac{1}{4}\right]+\left[1-\frac{1}{3}\right]\right\}$	M1		
	Obtains results.	$=\frac{1}{2}\left\{\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right\}$ (acf) $\Rightarrow S_\infty=\frac{3}{4}$	A1A1√	5	**[5]**

MATRICES 1

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Q1

4 The matrix **M** represents the sequence of two transformations in the x-y plane given by a rotation of 60° anticlockwise about the origin followed by a one-way stretch in the x-direction, scale factor d ($d \neq 0$).

(a) Find **M** in terms of d. [4]

(b) The unit square in the x-y plane is transformed by **M** onto a parallelogram of area $\frac{1}{2}d^2$ units^2.

Show that $d = 2$. [2]

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The matrix **N** is such that $\mathbf{MN} = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

(c) Find **N**. [3]

(d) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **MN**. [5]

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Q2

4 The matrices **A**, **B** and **C** are given by

$$\mathbf{A} = \begin{pmatrix} 2 & k & k \\ 5 & -1 & 3 \\ 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \quad \mathbf{C} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix},$$

where k is a real constant.

(a) Find **CAB**. [3]

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(b) Given that **A** is singular, find the value of k. [3]

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(c) Using the value of k from part **(b)**, find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

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Q3

4 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) The matrix **M** represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

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(b) Find the values of θ, for $0 \leqslant \theta \leqslant \pi$, for which the transformation represented by **M** has exactly one invariant line through the origin, giving your answers in terms of π. [9]

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Q4

1 **(a)** Give full details of the geometrical transformation in the x-y plane represented by the matrix $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. [1]

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Let $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}$.

(b) The triangle DEF in the x-y plane is transformed by **A** onto triangle PQR.

Given that the area of triangle DEF is 13 cm^2, find the area of triangle PQR. [2]

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(c) Find the matrix **B** such that $\mathbf{AB} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. [2]

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(d) Show that the origin is the only invariant point of the transformation in the x-y plane represented by **A**. [4]

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Q5

1 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, where a and b are positive constants.

(a) The matrix **M** represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

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The unit square in the x-y plane is transformed by **M** onto parallelogram $OPQR$.

(b) Find, in terms of a and b, the matrix which transforms parallelogram $OPQR$ onto the unit square. [2]

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It is given that the area of $OPQR$ is $2\,\text{cm}^2$ and that the line $x+3y=0$ is invariant under the transformation represented by **M**.

(c) Find the values of a and b. [5]

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Q6

4 The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}.$$

(a) Give full details of the geometrical transformation in the *x*-*y* plane represented by **A**. [1]

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(b) Give full details of the geometrical transformation in the *x*-*y* plane represented by **B**. [2]

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The triangle *DEF* in the *x*-*y* plane is transformed by **AB** onto triangle *PQR*.

(c) Show that the triangles *DEF* and *PQR* have the same area. [3]

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(d) Find the matrix which transforms triangle PQR onto triangle DEF. [2]

(e) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **AB**. [5]

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Q7

6 Let $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) The transformation in the x-y plane represented by $\mathbf{A}^{-1}$ transforms a triangle of area $30\,\text{cm}^2$ into a triangle of area $d\,\text{cm}^2$.

Find the value of d. [3]

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(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}.$$ [5]

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(c) The line $y = 2x$ is invariant under the transformation in the x-y plane represented by $\mathbf{A}^n\mathbf{B}$, where $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix}$.

Find the value of n. [5]

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Q8

4 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} k & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & -k \end{pmatrix},$$

where k is a real constant.

(a) Show that **A** is non-singular. [3]

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The matrices **B** and **C** are given by

$$\mathbf{B} = \begin{pmatrix} 0 & -3 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

It is given that $\mathbf{CAB} = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{3}{2} \end{pmatrix}$.

(b) Find the value of k. [3]

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(c) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

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MATRICES 1
MS

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Q1

Question	Answer	Marks	Guidance
4(a)	$\begin{pmatrix}\cos 60 & -\sin 60\\ \sin 60 & \cos 60\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$	**B1**	Rotation 60° anticlockwise about the origin.
	$\begin{pmatrix}d & 0\\ 0 & 1\end{pmatrix}$	**B1**	One-way stretch in the x-direction, scale factor d.
	$\mathbf{M}=\begin{pmatrix}d & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{d}{2} & -\frac{d\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$	**M1 A1**	Correct order.
		4	
4(b)	$d=\frac{1}{2}d^2$	**M1**	Uses value of $\det\mathbf{M}$.
	$d\neq 0\Rightarrow d=2$	**A1**	AG
		2	
4(c)	$\mathbf{M}^{-1}=\frac{1}{2}\begin{pmatrix}\frac{1}{2} & \sqrt{3}\\ -\frac{\sqrt{3}}{2} & 1\end{pmatrix}$	**B1 FT**	Inverse of *their* **M**.
	$\mathbf{N}=\mathbf{M}^{-1}\begin{pmatrix}1 & 1\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\frac{1}{2} & \sqrt{3}\\ -\frac{\sqrt{3}}{2} & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$	**M1**	Multiplies on the left by the inverse of *their* **M**.
	$=\frac{1}{4}\begin{pmatrix}1+\sqrt{3} & 1+\sqrt{3}\\ 1-\sqrt{3} & 1-\sqrt{3}\end{pmatrix}$	**A1**	CAO

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4(d)	$\begin{pmatrix}1 & 1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x+y\\ \frac{1}{2}x+\frac{1}{2}y\end{pmatrix}$	**B1**	Transforms $\begin{pmatrix}x\\ y\end{pmatrix}$ to $\begin{pmatrix}X\\ Y\end{pmatrix}$.
	$\frac{1}{2}x+\frac{1}{2}mx=m\left(x+mx\right)$	**M1 A1**	Uses $y=mx$ and $Y=mX$.
	$1+m=2m+2m^2\Rightarrow 2m^2+m-1=0$	**A1**	
	$y=\frac{1}{2}x$ and $y=-x$	**A1**	WWW
		5	

Q2

Question	Answer	Marks	Guidance
4(a)	$\begin{pmatrix}0 & 1 & 1\\ -1 & 2 & 0\end{pmatrix}\begin{pmatrix}2 & k & k\\ 5 & -1 & 3\\ 1 & 0 & \end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}0 & 1 & 1\\ -1 & 2 & 0\end{pmatrix}\begin{pmatrix}2+k & k\\ 8 & -1\\ 2 & 0\end{pmatrix}$ Or $\begin{pmatrix}0 & 1 & 1\\ -1 & 2 & 0\end{pmatrix}\begin{pmatrix}2 & k & k\\ 5 & -1 & 3\\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}6 & -1 & 4\\ 8 & -k-2 & -k+6\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 0\end{pmatrix}$	**M1 A1**	Multiplies two matrices correctly.
	$\begin{pmatrix}10 & -1\\ -k+14 & -k-2\end{pmatrix}$	**A1**	
		3	
4(b)	$2\begin{vmatrix}-1 & 3\\ 0 & 1\end{vmatrix}-k\begin{vmatrix}5 & 3\\ 1 & 1\end{vmatrix}+k\begin{vmatrix}5 & -1\\ 1 & 0\end{vmatrix}=0$ leading to $-2-2k+k=0$	**M1 A1**	Sets determinant equal to zero and forms linear equation.
	$k=-2$	**A1**	
		3	

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4(c)	$\begin{pmatrix}10 & -1\\16 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}10x-y\\16x\end{pmatrix}$	**M1**	Transforms $\begin{pmatrix}x\\y\end{pmatrix}$. Allow $q\begin{pmatrix}1\\m\end{pmatrix}$ where q is x, t or a nonzero number.
	$10x-mx=X$ and $16x=mX$	**M1 A1**	Uses $y=mx$ and $Y=mX$. Expect $16x=m(10x-mx)$.
	$16=10m-m^2$ $\quad[m^2-10m+16=0]$	**A1**	OE
	$y=2x$ and $y=8x$	**A1**	
		5	

Q3

Question	Answer	Marks	Guidance
4(a)	One-way stretch [scale factor 3 parallel to the x-axis] followed by a rotation, [anticlockwise centred at the origin, through an angle θ.]	**B2**	Award B1 if given in the wrong order.
		2	

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4(b)	$\mathbf{M}=\begin{pmatrix}3\cos\theta & -\sin\theta\\ 3\sin\theta & \cos\theta\end{pmatrix}$	**B1**	
	$\begin{pmatrix}3\cos\theta & -\sin\theta\\ 3\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}3x\cos\theta-y\sin\theta\\ 3x\sin\theta+y\cos\theta\end{pmatrix}$	**B1**	Transforms $\begin{pmatrix}x\\ y\end{pmatrix}$ to $\begin{pmatrix}X\\ Y\end{pmatrix}$.
		M1 A1	Uses $y=mx$ and $Y=mX$.
		A1	
		M1 A1	Sets discriminant equal to 0.
	$\tan\theta=\pm\frac{1}{\sqrt{3}}$ or $\cos\theta=\pm\frac{\sqrt{3}}{2}$	**M1**	Uses an appropriate trigonometric identity.
	$\theta=\frac{1}{6}\pi$ and $\theta=\frac{5}{6}\pi$	**A1**	Allow 30° and 150°
		9	

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Q4

Question	Answer	Marks	Guidance
1(a)	Enlargement, scale factor 6.	**B1**	
		1	
1(b)	$\det \mathbf{A} = 6 - 8 = -2$	**M1**	Finds $\det \mathbf{A}$.
	Area $= 2 \times 13 = 26\ \text{cm}^2$	**A1**	
		2	
1(c)	$\mathbf{A}^{-1} = -\frac{1}{2}\begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix}$	**M1**	Finds $\mathbf{A}^{-1}$.
	$\mathbf{B} = -3\begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -6 & 12 \\ 6 & -9 \end{pmatrix}$	**A1**	AEF Could be solved by equations.
		2	
1(d)	$\begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x+4y \\ 2x+2y \end{pmatrix}$	**B1**	Finds $\begin{pmatrix} X \\ Y \end{pmatrix}$
	$\left.\begin{matrix} 3x+4y=x \\ 2x+2y=y \end{matrix}\right\}$ leading to $\begin{cases} 2x+4y=0 \\ 2x+y=0 \end{cases}$	**M1**	Uses $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$ to form simultaneous equations.
	$y = -2x$ leading to $-6x = 0$ leading to $x = 0,\ y = 0$	**M1 A1**	Solves equations or states that $\det\begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix} \neq 0$, AG.
		4	

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Q5

Question	Answer	Marks	Guidance
1(a)	One-way stretch followed by a shear.	**B2**	Both named correctly. Award B1 if given in the wrong order.
		2	
1(b)	$\mathbf{M}^{-1} = \frac{1}{a}\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}$	**M1 A1**	
		2	
1(c)	$a = 2$	**B1**	
	$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ y \end{pmatrix}$	**B1**	Transforms $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$.
	$= \begin{pmatrix} ax - \frac{1}{3}bx \\ -\frac{1}{3}x \end{pmatrix}$	**M1**	Uses $x + 3y = 0$.
	$x = ax - \frac{1}{3}bx \Rightarrow 1 = a - \frac{1}{3}b$	**M1**	Uses that line is invariant (or $X + 3Y = 0$).
	$b = 3$	**A1**	
		5	

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Q6

Question	Answer	Marks	Guidance
4(a)	Reflection in the line $y = x$.	**B1**	Only mention reflection (and no other transformation).
		1	
4(b)	Rotation	**B1**	Writes 'rotation' or 'rotate' (and no other transformation)
	$\frac{1}{3}\pi$ **anticlockwise** about the origin.	**B1**	Or 60°
		2	
4(c)	$\mathbf{AB} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{pmatrix}$ or $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$	**M1**	Finds **AB** or uses product of determinants. Full marks here may be obtained by arguing that both reflection and rotation preserve [absolute] value of area and so also does their combination.
	$\det \mathbf{AB} = -1$	**B1**	
	Area of PQR = |-1| Area of DEF	**A1**	
		3	
4(d)	$(\mathbf{AB})^{-1} = -\begin{pmatrix} -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{pmatrix}$	**M1 A1**	or using $(\mathbf{AB})^{-1} = \mathrm{B}^{-1}\mathrm{A}^{-1}$
		2	
4(e)	$\begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x\sqrt{3} + \frac{1}{2}y \\ \frac{1}{2}x - \frac{1}{2}y\sqrt{3} \end{pmatrix}$	**B1**	Transforms $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$. Accept $t\begin{pmatrix} 1 \\ m \end{pmatrix}$ instead of $\begin{pmatrix} x \\ y \end{pmatrix}$.
	$\frac{1}{2} - \frac{1}{2}m\sqrt{3} = \frac{1}{2}m\sqrt{3} + \frac{1}{2}m^2$	**M1A1**	Uses $Y = mX$. OE. Allow working with $y = mx + c$ as long as $c = 0$ is stated explicitly.
	$1 - m\sqrt{3} = m\sqrt{3} + m^2 \Rightarrow m^2 + 2m\sqrt{3} - 1 = 0 \Rightarrow m = \pm 2 - \sqrt{3}$	**A1**	
	$y = (2 - \sqrt{3})x$ and $y + (2 + \sqrt{3})x = 0$	**A1**	OE
		5	

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Q7

Question	Answer	Marks
6(a)	$\det \mathbf{A}^{-1} = (\det \mathbf{A})^{-1} = \frac{1}{2}$	**M1 A1**
	$d = 15$	**A1**
		3
6(b)	$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^1 & 0 \\ 2^1 - 1 & 1 \end{pmatrix}$ so true when $n = 1$.	**B1**
	Assume that it is true for $n = k$, so $\mathbf{A}^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}$.	**B1**
	Then $\mathbf{A}^{k+1} = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 0 \\ 2(2^k - 1) + 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 1 & 1 \end{pmatrix}$	**M1A1**
	So, it is also true for $n = k + 1$. Hence, by induction, true for all positive integers.	**A1**
		5
6(c)	$\mathbf{A}^n\mathbf{B} = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix} = \begin{pmatrix} 2^n & 0 \\ 2^n + 32 & 0 \end{pmatrix}$	**M1A1**
	$\begin{pmatrix} 2^n & 0 \\ 2^n + 32 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2^n x \\ (2^n + 32)x \end{pmatrix}$	**B1**
	$(2^n + 32)x = 2^{n+1}x \Rightarrow 2^n = 32 \Rightarrow n = 5$	**M1 A1**
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Q8

Question	Answer	Marks
4(a)	$k\begin{vmatrix}-1 & -1\\ 1 & -k\end{vmatrix}+2\begin{vmatrix}0 & -1\\ 1 & 1\end{vmatrix}=0\Rightarrow k^2+k+2 \quad 0$	**M1 A1**
	$1-4(2)=-7<0\Rightarrow$ Non-singular	**A1**
		3
4(b)	$\begin{pmatrix}-3 & -1 & 1\\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix}k & 0 & 2\\ 0 & -1 & -1\\ 1 & 1 & -k\end{pmatrix}\begin{pmatrix}0 & -3\\ -1 & 3\\ 0 & 0\end{pmatrix}=\begin{pmatrix}-3 & -1 & 1\\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix}0 & -3k\\ 1 & -3\\ -1 & 0\end{pmatrix}$	**M1 A1**
	$=\begin{pmatrix}-2 & 9k+3\\ -1 & -3k-3\end{pmatrix}=\begin{pmatrix}-2 & -\frac{3}{2}\\ -1 & -\frac{3}{2}\end{pmatrix}\Rightarrow k=-\frac{1}{2}$	**A1**
		3
4(c)	$\begin{pmatrix}2 & \frac{3}{2}\\ 1 & \frac{3}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}2x+\frac{3}{2}y\\ x+\frac{3}{2}y\end{pmatrix}$	**B1**
	$x+\frac{3}{2}mx=m\left(2x+\frac{3}{2}mx\right)$	**M1 A1**
	$1+\frac{3}{2}m=2m+\frac{3}{2}m^2\Rightarrow 3m^2+m-2=0$	**A1**
	$y=-x$ and $3y-2x=0$	**A1**
		5

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POLAR COORDINATES

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Q1

5 The curve C has polar equation $r = a\cot\left(\frac{1}{3}\pi - \theta\right)$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{6}\pi$.

It is given that the greatest distance of a point on C from the pole is $2\sqrt{3}$.

(a) Sketch C and show that $a = 2$. [3]

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(b) Find the exact value of the area of the region bounded by C, the initial line and the half-line $\theta = \frac{1}{6}\pi$. [4]

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(c) Show that C has Cartesian equation $2\left(x+y\sqrt{3}\right)=\left(x\sqrt{3}-y\right)\sqrt{x^2+y^2}$. [3]

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Q2

5 The curve C has polar equation $r = \frac{1}{\pi - \theta} - \frac{1}{\pi}$, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Sketch C. [3]

(b) Show that the area of the region bounded by the half-line $\theta = \frac{1}{2}\pi$ and C is $\frac{3 - 4\ln 2}{4\pi}$. [6]

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Q3

6 The curve C has polar equation $r = 2\cos\theta(1+\sin\theta)$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Find the polar coordinates of the point on C that is furthest from the pole. [5]

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(b) Sketch C. [2]

(c) Find the area of the region bounded by C and the initial line, giving your answer in exact form. [6]

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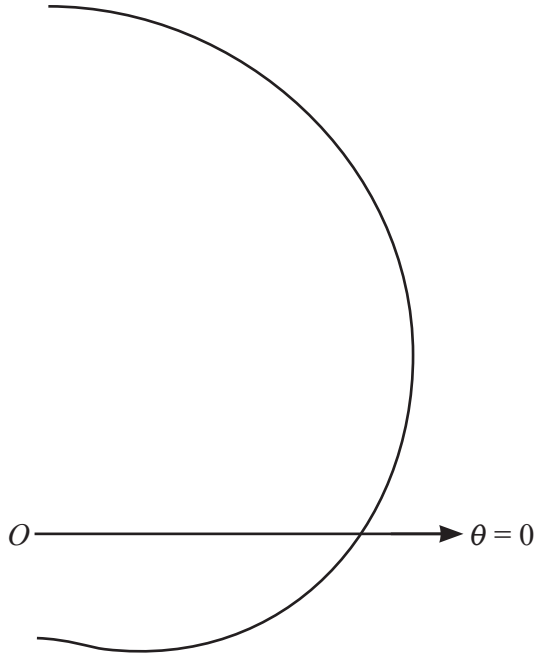
Q4

5 The curve C has polar equation $r = 3 + 2\sin\theta$, for $-\pi < \theta \leqslant \pi$.

(a) The diagram shows part of C. Sketch the rest of C on the diagram. [1]



The straight line l has polar equation $r\sin\theta = 2$.

(b) Add l to the diagram in part **(a)** and find the polar coordinates of the points of intersection of C and l. [5]

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(c) The region R is enclosed by C and l, and contains the pole.

Find the area of R, giving your answer in exact form. [6]

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Q5

7 **(a)** Show that the curve with Cartesian equation

$$(x^2+y^2)^{\frac{5}{2}} = 4xy(x^2-y^2)$$

has polar equation $r = \sin 4\theta$. [4]

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The curve C has polar equation $r = \sin 4\theta$, for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(b) Sketch C and state the equation of the line of symmetry. [3]

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(c) Find the exact value of the area of the region enclosed by C. [4]

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(d) Using the identity $\sin 4\theta \equiv 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$, find the maximum distance of C from the line $\theta = \frac{1}{2}\pi$. Give your answer correct to 2 decimal places. [6]

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Q6

5 The curve C has polar equation $r = \ln(1+\pi-\theta)$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. [3]

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(b) Using the substitution $u = 1+\pi-\theta$, or otherwise, show that the area of the region enclosed by C and the initial line is

$$\tfrac{1}{2}(1+\pi)\ln(1+\pi)\big(\ln(1+\pi)-2\big)+\pi. \qquad [6]$$

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(c) Show that, at the point of C furthest from the initial line,

$$(1+\pi-\theta)\ln(1+\pi-\theta)-\tan\theta=0$$

and verify that this equation has a root between 1.2 and 1.3. [5]

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Q7

7 The curve C_1 has polar equation $r = \theta\cos\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

$$2\theta\tan\theta - 1 = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

The curve C_2 has polar equation $r = \theta\sin\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(b) Find the polar coordinates of Q, giving your answers in exact form. [2]

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(c) Sketch C_1 and C_2 on the same diagram. [3]

(d) Find, in terms of π, the area of the region bounded by the arc OQ of C_1 and the arc OQ of C_2. [7]

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Q8

5 The curve C has polar equation $r = a\tan\theta$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(a) Sketch C and state the greatest distance of a point on C from the pole. [2]

(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = \frac{1}{4}\pi$. [4]

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(c) Show that C has Cartesian equation $y = \dfrac{x^2}{\sqrt{a^2 - x^2}}$. [3]

(d) Using your answer to part **(b)**, deduce the exact value of $\displaystyle\int_0^{\frac{1}{2}a\sqrt{2}} \frac{x^2}{\sqrt{a^2 - x^2}}\,\mathrm{d}x$. [2]

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Q9

OR

The curves C_1 and C_2 have polar equations, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, as follows:

$$C_1 : r = 2\left(e^{\theta} + e^{-\theta}\right),$$
$$C_2 : r = e^{2\theta} - e^{-2\theta}.$$

The curves intersect at the point P where $\theta = \alpha$.

(i) Show that $e^{2\alpha} - 2e^{\alpha} - 1 = 0$. Hence find the exact value of α and show that the value of r at P is $4\sqrt{2}$. [6]

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(ii) Sketch C_1 and C_2 on the same diagram. [3]

(iii) Find the area of the region enclosed by C_1, C_2 and the initial line, giving your answer correct to 3 significant figures. [5]

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Q10

11 Answer only **one** of the following two alternatives.

EITHER

The curve C_1 has polar equation $r^2 = 2\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

$$2\theta \tan \theta = 1$$

and verify that this equation has a root between 0.6 and 0.7. [5]

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The curve C_2 has polar equation $r^2 = \theta \sec^2 \theta$, for $0 \leqslant \theta < \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(ii) Find the exact value of θ at Q. [2]

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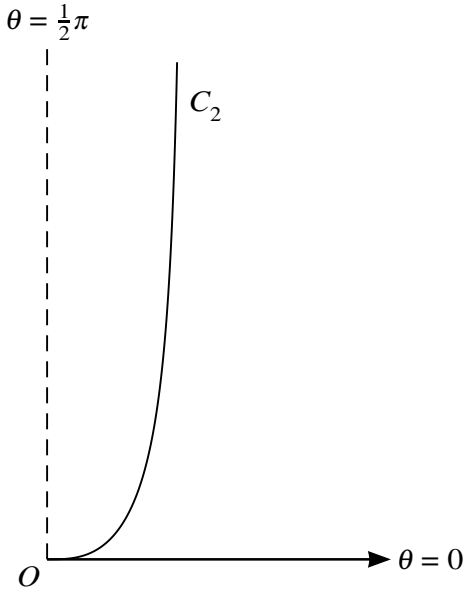
(iii) The diagram below shows the curve C_2. Sketch C_1 on this diagram. [2]



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(iv) Find, in exact form, the area of the region OPQ enclosed by C_1 and C_2. [5]

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Q11

2 The curve C has polar equation $r^2 = \ln(1 + \theta)$, for $0 \leqslant \theta \leqslant 2\pi$.

(i) Sketch C. [2]

(ii) Using the substitution $u = 1 + \theta$, or otherwise, find the area of the region bounded by C and the initial line, leaving your answer in an exact form. [5]

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Q12

3 The curve C has polar equation $r = \cos 2\theta$, for $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi$.

(i) Sketch C. [2]

(ii) Find the area of the region enclosed by C, showing full working. [3]

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(iii) Find a cartesian equation of C. [3]

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Q13

8 The curves C_1 and C_2 have polar equations, for $0 \leqslant \theta \leqslant \pi$, as follows:

$$C_1: r = a,$$
$$C_2: r = 2a|\cos\theta|,$$

where a is a positive constant. The curves intersect at the points P_1 and P_2.

(i) Find the polar coordinates of P_1 and P_2. [2]

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(ii) In a single diagram, sketch C_1, C_2 and their line of symmetry. [3]

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(iii) The region R enclosed by C_1 and C_2 is bounded by the arcs OP_1, P_1P_2 and P_2O, where O is the pole. Find the area of R, giving your answer in exact form. [5]

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Q14

9 The curve C has polar equation

$$r = 5\sqrt{(\cot \theta)},$$

where $0.01 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) Find the area of the finite region bounded by C and the line $\theta = 0.01$, showing full working. Give your answer correct to 1 decimal place. [3]

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Let P be the point on C where $\theta = 0.01$.

(ii) Find the distance of P from the initial line, giving your answer correct to 1 decimal place. [2]

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(iii) Find the maximum distance of C from the initial line. [3]

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(iv) Sketch C. [2]

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Q15

3 The curve C has polar equation $r = a\cos 3\theta$, for $-\frac{1}{6}\pi \leqslant \theta \leqslant \frac{1}{6}\pi$, where a is a positive constant.

(i) Sketch C. [2]

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(ii) Find the area of the region enclosed by C, showing full working. [3]

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(iii) Using the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$, find a cartesian equation of C. [3]

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Q16

11 The curve C has polar equation $r = a(1 + \sin\theta)$ for $-\pi < \theta \leqslant \pi$, where a is a positive constant.

(i) Sketch C. [2]

(ii) Find the area of the region enclosed by C. [4]

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(iii) Show that the length of the arc of C from the pole to the point furthest from the pole is given by

$$s = (\sqrt{2})a \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sqrt{(1 + \sin\theta)}\, \mathrm{d}\theta. \qquad [3]$$

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(iv) Show that the substitution $u = 1 + \sin\theta$ reduces this integral for s to $(\sqrt{2})a\int_0^2 \frac{1}{\sqrt{(2-u)}}\,\mathrm{d}u$. Hence evaluate s. [4]

Craftear

Q17

11 Answer only **one** of the following two alternatives.

EITHER

A curve C has polar equation $r = 2a\cos\left(2\theta + \frac{1}{2}\pi\right)$ for $0 \leqslant \theta < 2\pi$, where a is a positive constant.

(i) Show that $r = -2a\sin 2\theta$ and sketch C. [4]

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(ii) Deduce that the cartesian equation of C is

$$(x^2 + y^2)^{\frac{3}{2}} = -4axy.$$ [2]

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(iii) Find the area of one loop of C. [5]

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(iv) Show that, at the points (other than the pole) at which a tangent to C is parallel to the initial line,

$$2\tan\theta = -\tan 2\theta.$$ [3]

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Q18

OR

The polar equation of a curve C is $r = a(1 + \cos\theta)$ for $0 \leqslant \theta < 2\pi$, where a is a positive constant.

(i) Sketch C. [2]

(ii) Show that the cartesian equation of C is

$$x^2 + y^2 = a\left(x + \sqrt{(x^2 + y^2)}\right).$$ [2]

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(iii) Find the area of the sector of C between $\theta = 0$ and $\theta = \frac{1}{3}\pi$. [4]

Craftear

(iv) Find the arc length of C between the point where $\theta = 0$ and the point where $\theta = \frac{1}{3}\pi$. [5]

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Q19

4 A curve C has polar equation $r^2 = 8\,\text{cosec}\,2\theta$ for $0 < \theta < \frac{1}{2}\pi$. Find a cartesian equation of C. [3]

Sketch C. [2]

Determine the exact area of the sector bounded by the arc of C between $\theta = \frac{1}{6}\pi$ and $\theta = \frac{1}{3}\pi$, the half-line $\theta = \frac{1}{6}\pi$ and the half-line $\theta = \frac{1}{3}\pi$. [3]

[It is given that $\int \text{cosec}\,x\,\text{d}x = \ln\left|\tan\tfrac{1}{2}x\right| + c$.]

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[Turn over

7 A curve has polar equation $r = \dfrac{1}{1 - \cos\theta}$, for $0 < \theta < 2\pi$. Find, in the form $y^2 = \mathrm{f}(x)$, the cartesian equation of the curve. [3]

Hence sketch the curve, and shade the region whose area is given by $\frac{1}{2}\displaystyle\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{1}{(1-\cos\theta)^2}\,\mathrm{d}\theta$. [3]

Using the cartesian equation of the curve, find the area of this region. [3]

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5 The curves C_1 and C_2 have polar equations

$$C_1: \quad r = \frac{1}{\sqrt{2}}, \quad \text{for } 0 \leqslant \theta < 2\pi,$$
$$C_2: \quad r = \sqrt{\left(\sin \tfrac{1}{2}\theta\right)}, \quad \text{for } 0 \leqslant \theta \leqslant \pi.$$

Find the polar coordinates of the point of intersection of C_1 and C_2. [2]

Sketch C_1 and C_2 on the same diagram. [3]

Find the exact value of the area of the region enclosed by C_1, C_2 and the half-line $\theta = 0$. [4]

Craftear

Q22

2 The curve C has polar equation $r = e^{4\theta}$ for $0 \leqslant \theta \leqslant \alpha$, where α is measured in radians. The length of C is 2015. Find the value of α. [6]

Craftear

Q23

11 **OR**

The curve C has polar equation $r = a(1 - \cos\theta)$ for $0 \leqslant \theta < 2\pi$. Sketch C. [2]

Find the area of the region enclosed by the arc of C for which $\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$, the half-line $\theta = \frac{1}{2}\pi$ and the half-line $\theta = \frac{3}{2}\pi$. [5]

Show that

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2 = 4a^2 \sin^2\left(\tfrac{1}{2}\theta\right),$$

where s denotes arc length, and find the length of the arc of C for which $\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$. [7]

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[Turn over

5 The curve C has polar equation $r = a(1 + \sin\theta)$, where a is a positive constant and $0 \leqslant \theta < 2\pi$. Draw a sketch of C. [2]

Find the exact value of the area of the region enclosed by C and the half-lines $\theta = \frac{1}{3}\pi$ and $\theta = \frac{2}{3}\pi$. [4]

Craftear

4 The curve C has cartesian equation $x^2 + y^2{}^2 = 2a^2xy$, where a is a positive constant. Show that the polar equation of C is $r^2 = a^2 \sin 2\theta$. [3]

Sketch C for $-\pi < \theta \leqslant \pi$. [2]

Find the area enclosed by one loop of C. [2]

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8 A circle has polar equation $r = a$, for $0 \leqslant \theta < 2\pi$, and a cardioid has polar equation $r = a(1 - \cos\theta)$, for $0 \leqslant \theta < 2\pi$, where a is a positive constant. Draw sketches of the circle and the cardioid on the same diagram. [3]

Write down the polar coordinates of the points of intersection of the circle and the cardioid. [2]

Show that the area of the region that is both inside the circle and inside the cardioid is

$$(\tfrac{5}{4}\pi - 2)a^2.$$ [6]

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1 Find the area of the region enclosed by the curve with polar equation $r = 2(1 + \cos\theta)$, for $0 \leqslant \theta < 2\pi$. [4]

Craftear

10 Use the identity $2\sin P\cos Q \equiv \sin(P+Q) + \sin(P-Q)$ to show that

$$2\sin\theta\cos(\theta - \tfrac{1}{4}\pi) \equiv \cos(2\theta - \tfrac{3}{4}\pi) + \frac{1}{\sqrt{2}}. \quad [3]$$

A curve has polar equation $r = 2\sin\theta\cos(\theta - \frac{1}{4}\pi)$, for $0 \leqslant \theta \leqslant \frac{3}{4}\pi$. Sketch the curve and state the polar equation of its line of symmetry, justifying your answer. [3]

Show that the area of the region enclosed by the curve is $\frac{3}{8}(\pi + 1)$. [6]

Craftear

1 The curve C has polar equation $r = 2e^{\theta}$, for $\frac{1}{6}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. Find

(i) the area of the region bounded by the half-lines $\theta = \frac{1}{6}\pi$, $\theta = \frac{1}{2}\pi$ and C, [2]

(ii) the length of C. [3]

Craftear

10 The curve C has polar equation $r = 2\sin\theta(1 - \cos\theta)$, for $0 \leqslant \theta \leqslant \pi$. Find $\dfrac{\mathrm{d}r}{\mathrm{d}\theta}$ and hence find the polar coordinates of the point of C that is furthest from the pole. [5]

Sketch C. [2]

Find the exact area of the sector from $\theta = 0$ to $\theta = \frac{1}{4}\pi$. [6]

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4 The curve C has polar equation $r = 2 + 2\cos\theta$, for $0 \leqslant \theta \leqslant \pi$. Sketch the graph of C. [2]

Find the area of the region R enclosed by C and the initial line. [4]

The half-line $\theta = \frac{1}{5}\pi$ divides R into two parts. Find the area of each part, correct to 3 decimal places. [3]

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Q32

1 Find the cartesian equation corresponding to the polar equation $r = (\sqrt{2})\sec(\theta - \frac{1}{4}\pi)$. [3]

Sketch the the graph of $r = (\sqrt{2})\sec(\theta - \frac{1}{4}\pi)$, for $-\frac{1}{4}\pi < \theta < \frac{3}{4}\pi$, indicating clearly the polar coordinates of the intersection with the initial line. [2]

Craftear

5 The curve C has polar equation $r = 1 + 2\cos\theta$. Sketch the curve for $-\frac{2}{3}\pi \leqslant \theta < \frac{2}{3}\pi$. [2]

Find the area bounded by C and the half-lines $\theta = -\frac{1}{3}\pi$, $\theta = \frac{1}{3}\pi$. [4]

Craftear

11 **EITHER**

The curve C has cartesian equation

$$(x^2 + y^2)^2 = a^2(x^2 - y^2),$$

where a is a positive constant. Show that C has polar equation

$$r^2 = a^2 \cos 2\theta. \quad [2]$$

Sketch C for $-\pi < \theta \leqslant \pi$. [2]

Find the area of the sector between $\theta = -\frac{1}{4}\pi$ and $\theta = \frac{1}{4}\pi$. [3]

Find the polar coordinates of all points of C where the tangent is parallel to the initial line. [7]

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POLAR COORDINATES
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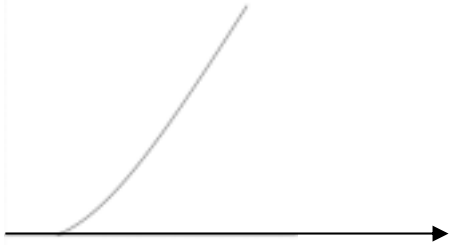
Q1

Question	Answer	Marks	Guidance
5(a)	$\theta = 0$ 	**B1 B1**	Initial line with correct position and shape of curve. SCB1 if correct shape with wrong starting position. Pole position must be clear.
	$a\cot\frac{1}{6}\pi = 2\sqrt{3} \Rightarrow a = 2$	**B1**	Substitutes $\theta = \frac{1}{6}\pi$. AG.
		3	
5(b)	$\frac{1}{2}\int_0^{\frac{1}{6}\pi} 4\cot^2\left(\frac{1}{3}\pi - \theta\right) \mathrm{d}\theta$	**M1**	Uses $\frac{1}{2}\int r^2 \,\mathrm{d}\theta$ with correct limits.
	$2\int_0^{\frac{1}{6}\pi} \mathrm{cosec}^2\left(\frac{1}{3}\pi - \theta\right) - 1 \,\mathrm{d}\theta = 2\left[\cot\left(\frac{1}{3}\pi - \theta\right) - \theta\right]_0^{\frac{1}{6}\pi}$	**M1 A1**	Uses $\cot^2\left(\frac{1}{3}\pi - \theta\right) = \mathrm{cosec}^2\left(\frac{1}{3}\pi - \theta\right) - 1$ and integrates.
	$2\left(\sqrt{3} - \frac{1}{6}\pi - \frac{1}{3}\sqrt{3}\right) = \frac{4}{3}\sqrt{3} - \frac{1}{3}\pi$	**A1**	OE, must be exact.
		4	
5(c)	$r = 2\dfrac{\cos\frac{\pi}{3}\cos\theta + \sin\frac{\pi}{3}\sin\theta}{\sin\frac{\pi}{3}\cos\theta - \cos\frac{\pi}{3}\sin\theta}$	**M1**	Uses the identities for $\cos(A-B)$ and $\sin(A-B)$. Or identity for tan $(A-B)$.
	$\sqrt{x^2+y^2} = 2\dfrac{r\cos\theta + \sqrt{3}r\sin\theta}{\sqrt{3}r\cos\theta - r\sin\theta}$	**M1**	Applies $r = \sqrt{x^2+y^2}$, $x = r\cos\theta$ and $y = r\sin\theta$.
	$2\left(x + y\sqrt{3}\right) = \left(x\sqrt{3} - y\right)\sqrt{x^2+y^2}$	**A1**	AG.
		3	

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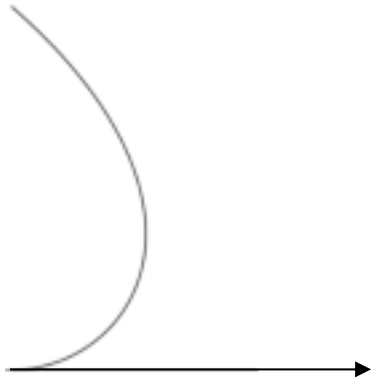
Q2

Question	Answer	Marks	Guidance
5(a)	 [O] $\theta = 0$]	**B1**	Approximately correct curve passing through the pole, O, with correct domain.
		B1	r strictly increasing over the domain 0 to $\frac{\pi}{2}$.
		B1	Correct form at O. Approx tangential to initial line at O.
		3	
5(b)	$\frac{1}{2}\int_0^{\frac{1}{2}\pi}\left(\frac{1}{\pi-\theta}-\frac{1}{\pi}\right)^2 \mathrm{d}\theta$	**M1**	Uses $\frac{1}{2}\int r^2 \,\mathrm{d}\theta$ with correct limits.
	$\frac{1}{2}\int_0^{\frac{1}{2}\pi}\frac{1}{(\pi-\theta)^2}-\frac{2}{\pi(\pi-\theta)}+\frac{1}{\pi^2}\,\mathrm{d}\theta$	**M1**	Expands.
	$\frac{1}{2}\left[\frac{1}{\pi-\theta}+\frac{2}{\pi}\ln(\pi-\theta)+\frac{\theta}{\pi^2}\right]_0^{\frac{1}{2}\pi}$	**M1 A1**	Integrates all terms to obtain correct form.
	$\frac{1}{2}\left(\frac{2}{\pi}+\frac{2}{\pi}\ln\frac{\pi}{2}+\frac{1}{2\pi}-\left(\frac{1}{\pi}+\frac{2}{\pi}\ln\pi\right)\right)=\frac{1}{2}\left(\frac{3}{2\pi}+\frac{2}{\pi}\ln\frac{1}{2}\right)$	**M1**	Substitute limits into an expression of the correct form and simplify the log terms.
	$=\frac{3-4\ln 2}{4\pi}$	**A1**	AG
		6	

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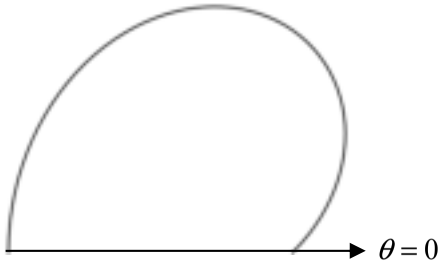
Q3

6(a)	$2\big(c(c)-(1+s)s\big)=0$	**M1**	Finds $\dfrac{\mathrm{d}r}{\mathrm{d}\theta}$ and sets equal to 0.
	$1-s-2s^2=0$	**A1**	
	$\sin\theta=\frac{1}{2}$ leading to θ $\frac{}{6}$		
	$\left(\frac{3}{2}\sqrt{3},\frac{1}{6}\pi\right)$	**A1**	Allow $\left(2.60,\frac{1}{6}\pi\right)$ 3sf
		5	
6(b)		**B1**	Correct shape. Tangential to $\theta=\dfrac{\pi}{2}$ at pole, *r* strictly increasing to $\dfrac{\pi}{6}$ then decreasing.
		B1	Correct position. All in first quadrant.
		2	

Question	Answer	Marks	Guidance
6(c)	$\frac{1}{2}\int_0^{\frac{1}{2}\pi}4c^2(1+s)^2\,\mathrm{d}\theta=2\int_0^{\frac{1}{2}\pi}c^2+2sc^2+s^2c^2\,\mathrm{d}\theta$	**M1 A1**	Uses $\frac{1}{2}\int r^2\,\mathrm{d}\theta$ with correct limits.
	$2\int_0^{\frac{1}{2}\pi}\frac{1}{2}\big(\cos2\theta+1\big)+2\sin\theta\cos^2\theta+\frac{1}{4}\sin^2 2\theta\;\mathrm{d}\theta$	**M1**	Uses double angle formulae on first and last terms.
	$\int_0^{\frac{1}{2}\pi}\cos2\theta+1+4\sin\theta\cos^2\theta+\frac{1}{4}-\frac{1}{4}\cos4\theta\;\mathrm{d}\theta$	**A1**	Obtains integrable form.
	$\left[\frac{5}{4}\theta+\frac{1}{2}\sin2\theta-\frac{4}{3}\cos^3\theta-\frac{1}{16}\sin4\theta\right]_0^{\frac{1}{2}\pi}$	**M1**	Integrates.
	$\frac{5}{8}\pi+\frac{4}{3}$	**A1**	
		6	

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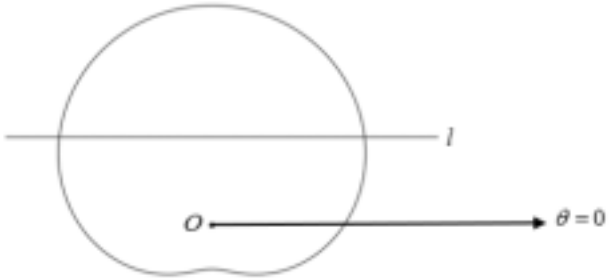
Q4

Question	Answer	Marks	Guidance
5(a)		**B1**	Correct symmetrical shape, closed loop.
		1	
5(b)	Line l parallel to initial line and correct side of pole.	**B1**	
$2 = 3\sin\theta + 2\sin^2\theta$		**M1**	Forms quadratic in $\sin\theta$. Or in r $r = 3 + \frac{4}{r}$
	$\sin\theta = \frac{1}{2}$	**M1**	Solves for $\sin\theta$.
	$\left(4, \frac{1}{6}\pi\right)$ $\left(4, \frac{5}{6}\pi\right)$	**A1** **A1**	**SC1** For finding both angles correctly
		5	

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5(c)	$2\times\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}}(3+2\sin\theta)^2\,\mathrm{d}\theta$	**M1**	Finds the part of the required area enclosed by the curved outer edge and two line segments from the pole. Limits must be correct.
	$\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}}9+12\sin\theta+2(1-\cos 2\theta)\,\mathrm{d}\theta$	**M1**	Uses double angle formula and integrates.
	$\left[11\theta-12\cos\theta-\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}=\frac{22}{3}\pi-\frac{13}{2}\sqrt{3}$	**A1 A1**	
	$\frac{22}{3}\pi-\frac{13}{2}\sqrt{3}+(4\cos\frac{\pi}{6})\times 2$	**M1**	Adds area of triangle.
	$\frac{22}{3}\pi-\frac{5}{2}\sqrt{3}$	**A1**	
		6	

Q5

Question	Answer	Marks	Guidance
7(a)	$r^5=4xy\left(x^2-y^2\right)$	**B1**	Uses $r^2=x^2+y^2$.
	$r^5=4r^4\cos\theta\sin\theta\left(\cos^2\theta-\sin^2\theta\right)$	**B1**	Uses $x=r\cos\theta$ and $y=r\sin\theta$.
	$r=2\sin 2\theta\cos 2\theta$	**M1**	Applies at least one double angle formula.
	$r=\sin 4\theta$	**A1**	Applies both double angle formulae, AG.
		4	

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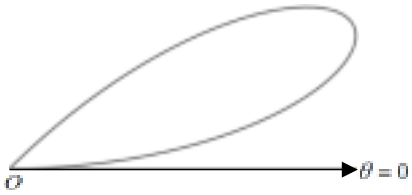
7(b)		**B1**	Initial line drawn and one loop in the first quadrant.
		B1	Correct shape at extremities.
	$\theta = \frac{1}{8}\pi$	**B1**	States the equation of the line of symmetry.
		3	
7(c)	$\frac{1}{2}\int_0^{\frac{1}{4}\pi} \sin^2 4\theta \, \mathrm{d}\theta$	**M1**	Uses $\frac{1}{2}\int r^2 \, \mathrm{d}\theta$ with correct limits.
	$= \frac{1}{4}\int_0^{\frac{1}{4}\pi} 1 - \cos 8\theta \, \mathrm{d}\theta = \frac{1}{4}\left[\theta - \frac{1}{8}\sin 8\theta\right]_0^{\frac{1}{4}\pi}$	**M1 A1**	Applies double angle formula and integrates.
	$= \frac{1}{16}\pi$	**A1**	
		4	
7(d)	$x = 4\left(\sin\theta\cos^4\theta - \sin^3\theta\cos^2\theta\right)$	**B1**	Uses $x = r\cos\theta$.
	$-4\sin^2\theta\cos^3\theta + \cos^5\theta + 2\sin^4\theta\cos\theta - 3\sin^2\theta\cos^3\theta = 0$	**M1**	Differentiates and sets equal to 0.
	$\cos\theta\left(2\sin^4\theta - 7\sin^2\theta\cos^2\theta + \cos^4\theta\right) = 0$	**A1**	
	$\cos\theta = 0$ or $2\tan^4\theta - 7\tan^2\theta + 1 = 0$	**M1**	Forms quadratic in $\tan^2\theta$. Must see consideration of $\cos\theta = 0$.
	$\tan^2\theta = \frac{1}{4}\left(7 \pm \sqrt{41}\right) \Rightarrow \theta = \pm 0.369, \pm 1.071$	**B1**	Allow use of decimals.
	$\theta = 0.369 \Rightarrow x = 0.93$ (or $\theta = -0.369 \Rightarrow \lvert x \rvert = 0.93$)	**A1**	Substituting $\theta = \pm 0.369$ gives maximum value of $\lvert x \rvert$.
		6	

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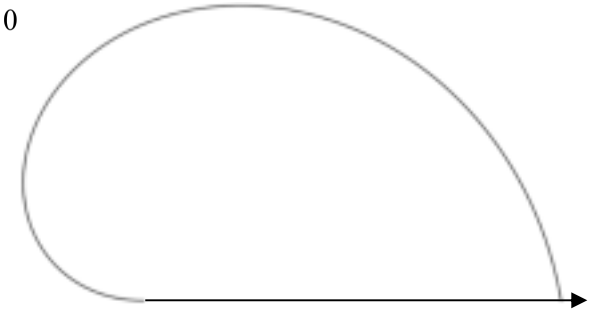
Q6

Question	Answer	Marks	Guidance
5(a)	$\theta = 0$ 	**B1**	Correct shape, r decreasing.
		B1	Section $\frac{1}{2}\pi \le \theta \le \pi$ correct, becoming tangential at $\theta = \pi$.
	$(\ln(1+\pi), 0)$	**B1**	May be seen on their diagram. Allow (1.42, 0)
		3	
5(b)	$\frac{1}{2}\int_0^{\pi} \ln^2(1+\pi-\theta)\,\mathrm{d}\theta = \frac{1}{2}\int_1^{1+\pi} \ln^2 u\,\mathrm{d}u$	**M1**	Uses correct formula with correct limits.
	$= \left[\frac{1}{2}u\ln^2 u\right]_1^{1+\pi} - \int_1^{1+\pi} \ln u\,du$	**M1 A1**	Integrates by parts once.
	$= \left[\frac{1}{2}u\ln^2 u\right]_1^{1+\pi} - \left(\left[u\ln u\right]_1^{1+\pi} - \int_1^{1+\pi} 1\,\mathrm{d}u\right)$	**M1**	Integrates by parts again (or uses known result for integral of $\ln u$)
	$= \left[\frac{1}{2}u\ln^2 u - u\ln u + u\right]_1^{1+\pi}$	**A1**	
	$\frac{1}{2}(1+\pi)\ln^2(1+\pi) - (1+\pi)\ln(1+\pi) + \pi =$ $\frac{1}{2}(1+\pi)\ln(1+\pi)(\ln(1+\pi)-2) + \pi$	**A1**	AG
		6	
5(c)	$y = \ln(1+\pi-\theta)\sin\theta$	**B1**	Uses $y = r\sin\theta$
	$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \ln(1+\pi-\theta)\cos\theta - \frac{\sin\theta}{1+\pi-\theta} = 0$	**M1 A1**	Sets derivative equal to zero.
	$\cos\theta \neq 0 \Rightarrow (1+\pi-\theta)\ln(1+\pi-\theta) - \tan\theta = 0$	**A1**	AG
	$(1+\pi-1.2)\ln(1+\pi-1.2) - \tan 1.2 = 0.602$ and $(1+\pi-1.3)\ln(1+\pi-1.3) - \tan 1.3 = -0.634$	**B1**	Shows sign change. Values correct to 2sf should be shown.
		5	

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Q7

Question	Answer	Marks
7(a)	$x = \theta\cos^2\theta$	**B1**
	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -2\theta\cos\theta\sin\theta + \cos^2\theta = 0$	**M1 A1**
	$\cos\theta \neq 0 \Rightarrow 2\theta\tan\theta - 1 = 0$	**A1**
	$2(0.6)\tan 0.6 - 1 = -0.179$ and $2(0.7)\tan 0.7 - 1 = 0.179$	**B1**
		5
7(b)	$\theta\cos\theta = \theta\sin\theta \Rightarrow \tan\theta = 1$	**M1**
	$\left(\frac{1}{8}\pi\sqrt{2}, \frac{1}{4}\pi\right)$	**A1**
		2
7(c)	(**B1** for initial line drawn and C_1 correct, **B1** for C_2 correct, **B1** for intersection correct)	**B1** **B1** **B1**
		3

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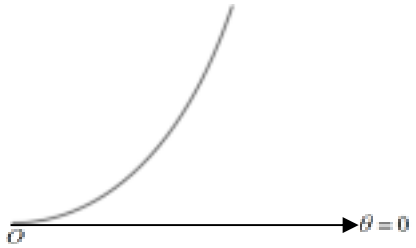
7(d)	$\frac{1}{2}\int_0^{\frac{1}{4}\pi}\theta^2\left(\cos^2\theta-\sin^2\theta\right)\mathrm{d}\theta$	**M1**
	$=\frac{1}{2}\int_0^{\frac{1}{4}\pi}\theta^2\cos 2\theta\,\mathrm{d}\theta$	**M1**
	$=\frac{1}{2}\left(\left[\frac{1}{2}\theta^2\sin 2\theta\right]_0^{\frac{1}{4}\pi}-\int_0^{\frac{1}{4}\pi}\theta\sin 2\theta\,\mathrm{d}\theta\right)$	**M1 A1**
	$=\frac{1}{2}\left(\left[\frac{1}{2}\theta^2\sin 2\theta\right]_0^{\frac{1}{4}\pi}-\left[-\frac{1}{2}\theta\cos 2\theta\right]_0^{\frac{1}{4}\pi}-\frac{1}{2}\int_0^{\frac{1}{4}\pi}\cos 2\theta\,\mathrm{d}\theta\right)$	**M1**
	$=\frac{1}{4}\left[\theta^2\sin 2\theta+\theta\cos 2\theta-\frac{1}{2}\sin 2\theta\right]_0^{\frac{1}{4}\pi}$	**A1**
	$=\frac{1}{64}\pi^2-\frac{1}{8}$	**A1**
		7

Q8

Question	Answer	Marks
5(a)		**B1**
	Maximum distance of C from the pole is a.	**B1**
		2

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5(b)	$\frac{1}{2}a^2\int_0^{\frac{1}{4}\pi}\tan^2\theta\,\mathrm{d}\theta$	**M1**
	$=\frac{1}{2}a^2\int_0^{\frac{1}{4}\pi}\sec^2\theta-1\,\mathrm{d}\theta=\frac{1}{2}a^2\left[\tan\theta-\theta\right]_0^{\frac{1}{4}\pi}$	**M1 A1**
	$=\frac{1}{2}a^2\left(1-\frac{1}{4}\pi\right)$	**A1**
		4
5(c)	$\sqrt{x^2+y^2}=a\frac{y}{x}$	**B1**
	$x^2\left(x^2+y^2\right)=a^2y^2\Rightarrow y^2\left(a^2-x^2\right)=x^4$	**M1**
	$y=\frac{x^2}{\sqrt{\left(a^2-x^2\right)}}$ **AG**	**A1**
		3
5(d)	$\frac{1}{2}\left(a\cos\frac{1}{4}\pi\right)\left(a\sin\frac{1}{4}\pi\right)-\frac{1}{2}a^2\left(1-\frac{1}{4}\pi\right)=\frac{1}{4}a^2\left(\frac{1}{2}\pi-1\right)$ (**A1 FT** *their* **(b)**)	**M1 A1 FT**
		11

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Q9

Question	Answer	Marks	Guidance
11O(i)	$e^{2\alpha} - e^{-2\alpha} = 2\left(e^{\alpha} + e^{-\alpha}\right) \Rightarrow e^{\alpha} - e^{-\alpha} = 2$	**M1**	Sets equations equal and divides by $e^{\alpha} + e^{-\alpha}$.
	$e^{2\alpha} - 2e^{\alpha} - 1 = 0 \Rightarrow e^{\alpha} = 1 + \sqrt{2}$	**M1 A1**	Forms quadratic in e^{α}, AG.
	$\alpha = \ln\left(1 + \sqrt{2}\right)$	**A1**	Must be exact.
	$r = 2\left(1 + \sqrt{2} + \sqrt{2} - 1\right) = 4\sqrt{2}$	**M1 A1**	Substitutes to find r.
		6	
11O(ii)		**B1**	C_1 has correct shape.
		B1	C_2 has correct shape.
		B1	Intersection points positioned correctly.
		3	

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Question	Answer	Marks	Guidance
11O(iii)	$2\int_0^{\ln(1+\sqrt{2})}\left(e^{\theta}+e^{-\theta}\right)^2\,d\theta-\frac{1}{2}\int_0^{\ln(1+\sqrt{2})}\left(e^{2\theta}-e^{-2\theta}\right)^2\,d\theta$ $=\int_0^{\ln(1+\sqrt{2})}5+2e^{2\theta}+2e^{-2\theta}-\frac{1}{2}e^{4\theta}-\frac{1}{2}e^{-4\theta}\,d\theta$	**M1 A1**	Uses $\frac{1}{2}\int r^2\,d\theta$ to formulate correct area.
	$=\left[5\theta+e^{2\theta}-e^{-2\theta}-\frac{1}{8}e^{4\theta}+\frac{1}{8}e^{-4\theta}\right]_0^{\ln(1+\sqrt{2})}$	**M1 A1**	Expands and integrates.
	$=5\ln(1+\sqrt{2})+\left(1+\sqrt{2}\right)^2-\left(1+\sqrt{2}\right)^{-2}-\frac{1}{8}\left(\left(1+\sqrt{2}\right)^4-\left(1+\sqrt{2}\right)^{-4}\right)=5.82$	**A1**	
		5	

Q10

Question	Answer	Marks	Guidance
11E(i)	$x=\sqrt{2}\theta^{\frac{1}{2}}\cos\theta$	**M1**	Uses $x=r\cos\theta$.
	$\frac{d}{d\theta}\left(\sqrt{2}\theta^{\frac{1}{2}}\cos\theta\right)=\sqrt{2}\left(-\theta^{\frac{1}{2}}\sin\theta+\frac{1}{2}\theta^{-\frac{1}{2}}\cos\theta\right)=0$	**M1 A1**	Sets derivative of $r\cos\theta$ equal to zero.
	$-\theta^{\frac{1}{2}}\sin\theta+\frac{1}{2}\theta^{-\frac{1}{2}}\cos\theta=0\Rightarrow\cos\theta=2\theta\sin\theta\Rightarrow 2\theta\tan\theta=1$	**A1**	AG
	$2(0.6)\tan(0.6)-1=-0.179$ and $2(0.7)\tan(0.7)-1=0.179$	**B1**	Shows sign change.
		5	

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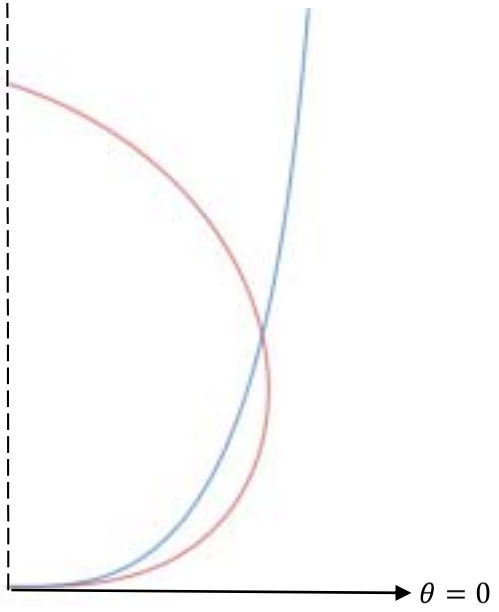
11E(ii)	$2\theta = \theta\sec^2\theta \Rightarrow \theta^{\frac{1}{2}}\left(\sec\theta - \sqrt{2}\right) = 0 \Rightarrow \theta = \frac{\pi}{4}$	**M1 A1**	Finds value of θ.
		2	
11E(iii)		**B1**	Correct shape.
		B1	Intersection correct.
		2	
11E(iv)	$\frac{1}{2}\int_0^{\frac{\pi}{4}} 2\theta \quad \theta - \frac{1}{2}\int_0^{\frac{\pi}{4}} \theta\sec^2\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \theta\left(2 - \sec^2\theta\right)\mathrm{d}\theta$	**M1**	Forms correct integral.
	$\frac{1}{2}\left[\theta\left(2\theta - \tan\theta\right)\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(2\theta - \tan\theta\right)\mathrm{d}\theta$	**M1 A1**	Integrates by parts.
	$\frac{1}{2}\left[\theta\left(2\theta - \tan\theta\right)\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\left[\theta^2 + \ln\cos\theta\right]_0^{\frac{\pi}{4}}$	**A1**	
	$\frac{\pi}{8}\left(\frac{\pi}{2} - 1\right) - \frac{1}{2}\left(\left(\frac{\pi}{4}\right)^2 + \frac{1}{2}\ln 2\right) = \frac{1}{4}\ln 2 + \frac{\pi}{8}\left(\frac{\pi}{4} - 1\right).$	**A1**	AEF, must be exact.
		5	

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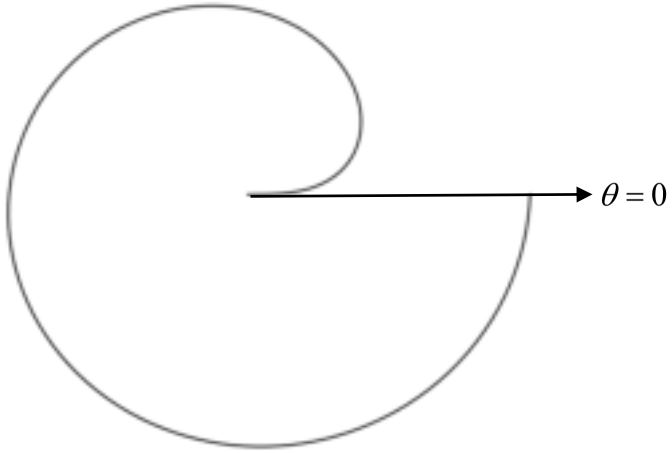
Q11

Question	Answer	Marks	Guidance
2(i)		**B1**	Correct shape, domain and orientation.
		B1	The initial line is tangential to C at the pole.
		2	
2(ii)	$A = \frac{1}{2}\int_0^{2\pi} \ln(1+\theta)\,\mathrm{d}\theta$	**M1**	States $\frac{1}{2}\int r^2\,\mathrm{d}\theta$ with correct expression and limits.
	$\int_0^{2\pi} (\quad) \qquad \int_1^{2\pi+1}$	**M1**	Applies given substitution correctly, changes their limits.
	$\int_1^{2\pi+1} \ln u\ \mathrm{d}u = \left[u \ln u\right]_1^{2\pi+1} - \left[u\right]_1^{2\pi+1}$	**M1 A1**	Integrates $\ln u$ (by parts or otherwise).
	$A = \left(\pi + \frac{1}{2}\right)\ln(2\pi+1) - \pi$	**A1**	AEF, must be exact.
		5	

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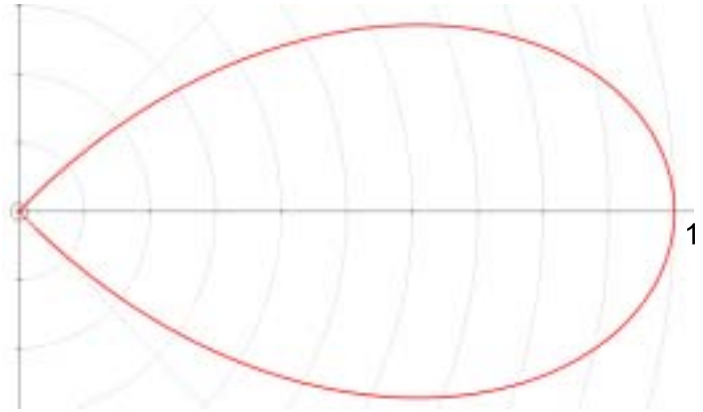
Q12

Question	Answer	Marks	Guidance
3(i)		**B1**	Correct position including label of 1 on initial line, and symmetric about initial
		B1	Single correct loop.
		2	
3(ii)	$\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} cos^2 2\theta d\theta$	**M1**	Correct integral
	$=\frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 4\theta + 1 d\theta =$	**M1**	Correct use of double angle formula
	$\frac{1}{4}\left[\frac{1}{4}\sin 4\theta + \theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{8} = 0.393$	**A1**	
		3	
3(iii)	$r = \cos^2\theta - \sin^2\theta$ OE	**B1**	Uses trig identity
	Thus $\left(x^2+y^2\right)^{\frac{3}{2}} = x^2 - y^2$	**M1 A1**	Uses $x = r\cos\theta$ or $y = r\sin\theta$ *or both* , AEF
		3	

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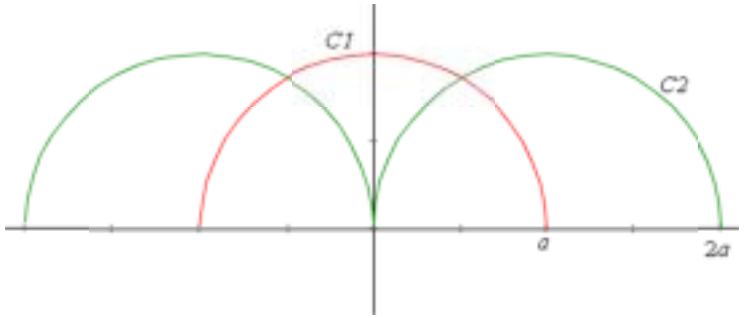
Q13

Question	Answer	Marks	Guidance
8(i)	$a = 2a\lvert\cos\theta\rvert \Rightarrow \cos\theta = \pm\frac{1}{2}$	**M1**	Eliminates r.
	$\left(a, \frac{\pi}{3}\right)$ and $\left(a, \frac{2\pi}{3}\right)$	**A1**	Both points needed for A1.
8(ii)		**B1**	Semicircle for C1 including r = a.
		B1	Half of C2 including r = 2a.
		B1	Other half of C2 and line of symmetry.
8(iii)	$4a^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2\theta \, d\theta$	**M1**	Finds area of segment OP_1 and OP_2 of C_2
	$= 2a^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos 2\theta + 1 \, d\theta$	**M1**	Uses $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$
	$= 2a^2\left[\frac{1}{2}\sin 2\theta + \theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = 2a^2\left(\frac{\pi}{2} - \left(\frac{\sqrt{3}}{4} + \frac{\pi}{3}\right)\right) = a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$	**A1**	Integrates correctly.
	Area $= \frac{\pi a^2}{6} - a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = -\frac{\pi a^2}{6} + \frac{a^2\sqrt{3}}{2}$	**M1** **A1FT**	M1 for subtracting 'their' OP_1P_2 from $\frac{\pi a^2}{6}$
		10	

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Q14

Question	Answer	Marks	Guidance
9(i)	$\frac{25}{2}\int_{0.01}^{\frac{\pi}{2}} \cot\theta \,\mathrm{d}\theta$	**M1**	Uses $\frac{1}{2}\int r^2 \,\mathrm{d}\theta$.
	$= \frac{25}{2}\left[\ln\sin\theta\right]_{0.01}^{\frac{\pi}{2}}$	**A1**	
	$= -\frac{25}{2}\ln\sin 0.01 \approx 57.6$	**A1**	
		3	
9(ii)	$y = 5\cos^{\frac{1}{2}}\theta \sin^{\frac{1}{2}}\theta = \frac{5}{\sqrt{2}}\sin^{\frac{1}{2}} 2\theta$	**M1**	Uses $y = r\sin\theta$.
	$\theta = 0.01 \Rightarrow y \approx 0.5$	**A1**	
		2	
9(iii)	$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{5}{\sqrt{2}}\sin^{-\frac{1}{2}} 2\theta\cos 2\theta = 0$ or $\max\left(\sin 2\theta\right) = 1$	**M1 A1**	Sets $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0$ or considers max (AEF).
	$\Rightarrow y = \frac{5\sqrt{2}}{2}$ $\left(= 3.54\right)$	**A1**	
		3	

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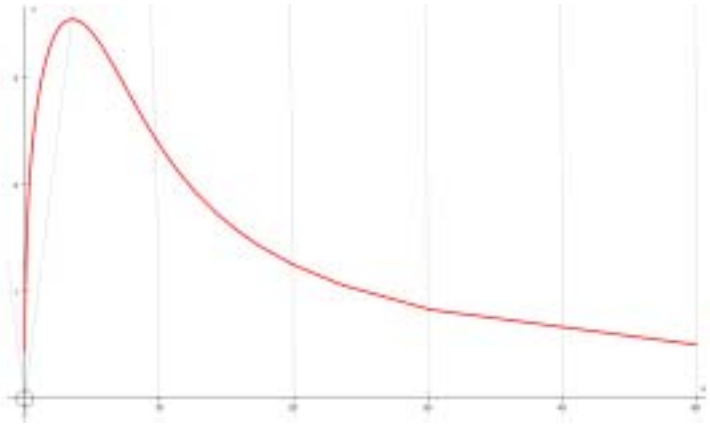
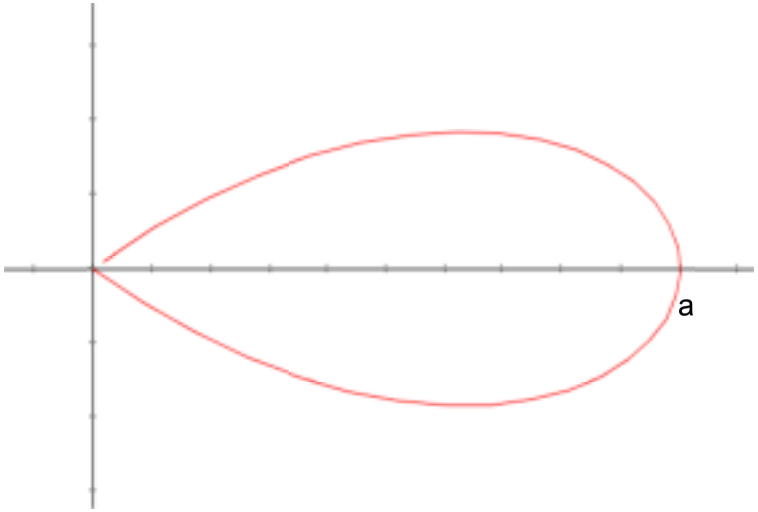
9(iv)		**B1**	Intersecting the initial line only when $x = 0$ and $y = 0$.
		B1	Correct shape.
		2	

Q15

Question	Answer	Marks	Guidance
3(i)		**B1**	Just one loop, correct shape at extremities
		B1	Correct position including (a, 0) labelled or in table.
		2	

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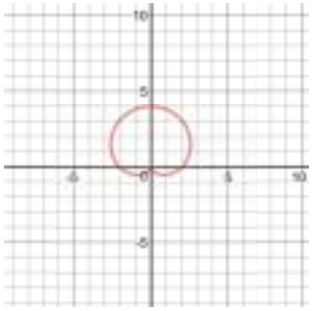
	$\frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} a^2 cos^2 3\theta \, d\theta$		
	$\frac{a^2}{4}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (\cos 6\theta + 1) d\theta$	**M1**	Using double angle formula correctly
	$= \frac{a^2}{4}\left[\frac{1}{6}\sin 6\theta + \theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{\pi a^2}{12}$	**A1**	
		3	
3(iii)	$r = a\cos\theta\left(4\cos^2\theta - 3\right) \Rightarrow r = a\left(\frac{x}{r}\right)\left(4\left(\frac{x}{r}\right)^2 - 3\right)$	**B1**	Uses $x = r\cos\theta$ **and** $x^2 + y^2 = r^2$.
	$\Rightarrow r^4 = ax\left(4x^2 - 3r^2\right) \Rightarrow \left(x^2 + y^2\right)^2 = ax\left(4x^2 - 3\left(x^2 + y^2\right)\right)$	**M1**	For eliminating θ
	$\Rightarrow \left(x^2 + y^2\right)^2 = ax\left(x^2 - 3y^2\right)$	**A1**	Any equivalent cartesian form without fractions.
		3	

Q16

Question	Answer	Marks	Guidance
11(i)		**B1**	Sketch of a cardioid
		B1	Correct orientation and labelled
	Total:	**2**	

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11(ii)	$A=\frac{a^2}{2}\int_{-\pi}^{\pi}\left(1+2\sin\theta+\sin^2\theta\right)\mathrm{d}\theta$	**M1**	
	$=\left(\frac{a^2}{2}\right)\int_{-\pi}^{\pi}\left(1+2\sin\theta+\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$	**M1**	
	$=\left(\frac{a^2}{2}\right)\left[\frac{3\theta}{2}-2\cos\theta-\frac{1}{4}\sin 2\theta\right]_{-\pi}^{\pi}=\frac{3\pi a^2}{2}$	**M1A1**	
	Total:	**4**	
11(iii)	Show that when r=0, $\theta=-\frac{\pi}{2}$, when r=2a $\theta=\frac{\pi}{2}$ and that $\frac{\mathrm{d}r}{\mathrm{d}\theta}=a\cos\theta$	**B1**	
	$s=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{a^2\left(1+2\sin\theta+a^2\sin^2\theta\right)+a^2\cos^2\theta}\ \mathrm{d}\theta=\sqrt{2a}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sqrt{1+\sin\theta}\ \mathrm{d}\theta$	**M1A1**	Uses correct formula for arc length; AG
	Total:	**3**	
11(iv)	$u=1+\sin\theta\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}\theta}=\cos\theta=\sqrt{1-(u-1)^2}=\sqrt{2u-u^2}$	**M1**	
	$so\ s=\sqrt{2a}\int_0^2\frac{1}{\sqrt{2-u}}\mathrm{d}u$ AG	**A1**	Including limits
	$=\sqrt{2a}\left[-2(2-u)^{\frac{1}{2}}\right]_0^2=4a$	**M1A1**	
	Total:	**4**	

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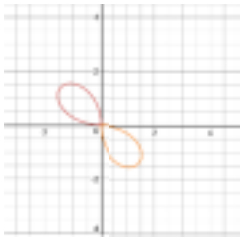
Q17

Question	Answer	Marks	Guidance
11E(i)	$r = 2a\left(\cos 2\theta \cos\frac{1}{2}\pi - \sin 2\theta \sin\frac{1}{2}\pi\right) = 2a\left(0 - \sin 2\theta.1\right) = -2a\sin 2\theta$	**M1A1**	AG
		**B1**	Loops in 2nd and 4th quadrant
		B1	Symmetry and shape correct
	Total:	**4**	
11E(ii)	$r = -4a\sin\theta\cos\theta = -4a.\frac{y}{r}.\frac{x}{r} \Rightarrow r^3 = -4axy \Rightarrow \left(\left(x^2 + y^2\right)^{\frac{3}{2}} = -4axy\right)$	**M1A1**	AG
	Total:	**2**	
11E(iii)	Area of one loop is $\frac{1}{2}\int_{\frac{1}{2}\pi}^{\pi} 4a^2\sin^2 2\theta \,\mathrm{d}\theta = a^2\int_{\frac{1}{2}\pi}^{\pi} 2\sin^2 2\theta \,\mathrm{d}\theta$	**M1**	OE
	$= a^2\int_{\frac{1}{2}\pi}^{\pi}\left(1 - \cos 4\theta\right)\mathrm{d}\theta$	**M1A1**	
	$= a^2\left[\theta - \frac{1}{4}\sin 4\theta\right]_{\frac{1}{2}\pi}^{\pi} \quad = \frac{1}{2}\pi a^2$	**M1A1**	
	Total:	**5**	

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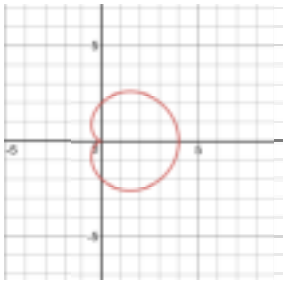
11E(iv)	$y = r\sin\theta = -2a\sin 2\theta\sin\theta$	**B1**	
	$\frac{dy}{d\theta} = -2a\left(2\cos 2\theta\sin\theta + \sin 2\theta\cos\theta\right) = 0$	**M1**	
	$2\cos 2\theta\sin\theta = -\sin 2\theta\cos\theta \quad (\Rightarrow 2\tan\theta = -\tan 2\theta)$	**A1**	AG
		Total: **3**	

Q18

Question	Answer	Marks	Guidance
11OR		**B1**	Closed curve starting and ending at pole, in approximately correct location.
		B1	Cardioid with indication of correct scale.
		2	
11OR(ii)	$r = a\left(1+\cos\theta\right) \Rightarrow \sqrt{x^2+y^2} = a\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)$	**M1**	
	$x^2+y^2 = a(x+\sqrt{(x^2+y^2)})$	**A1**	Substitutes for r **and** cos(θ)
		2	

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11OR(iii)	Sector area $= \frac{a^2}{2}\int_0^{\frac{1}{3}\pi}\left(1+2\cos\theta+\cos^2\theta\right)\mathrm{d}\theta$ $= \frac{a^2}{2}\int_0^{\frac{1}{3}\pi}\left(\frac{3}{2}+2\cos\theta+\frac{\cos 2\theta}{2}\right)\mathrm{d}\theta$	**M1A1**	
	$= \frac{a^2}{2}\left[\frac{3\theta}{2}+2\sin\theta+\frac{\sin 2\theta}{4}\right]_0^{\frac{1}{3}\pi}$	**M1**	
	$= \frac{a^2}{16}\left(4\pi+9\sqrt{3}\right)$	**A1**	
		4	
11OR(iv)	Arc length $= \int_0^{\frac{1}{3}\pi}\sqrt{a^2\left(1+2\cos\theta+\cos^2\theta\right)+a^2\left(-\sin\theta\right)^2}\,\mathrm{d}\theta$	**M1A1**	
	$= a\int_0^{\frac{1}{3}\pi}\sqrt{2+2\cos\theta}\,\mathrm{d}\theta$	**A1**	
	$= a\left[4\sin\frac{\theta}{2}\right]_0^{\frac{1}{3}\pi} = 2a$	**M1A1**	
		5	

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Q19

4	Using $x = r\cos\theta$ **and** $y = r\sin\theta$	**B1**
	$r^2 = 8\,\text{cosec}\,2\theta \Rightarrow r^2 = \dfrac{4}{\sin\theta\cos\theta}$	**M1**
	$\Rightarrow r\cos\theta . r\sin\theta = 4 \Rightarrow xy = 4$ (in simple form)	**A1** [3]
	Sketch: Curve in 1st quadrant with correct concavity, asymptotic to **both** axes.	**B1B1** [2]
	$\dfrac{1}{2}\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 8\,\text{cosec}\,2\theta\,\text{d}\theta = \left[2\ln\left\lvert\tan\theta\right\rvert\right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}$	**M1A1**
	$= 2\left\{\ln\left\lvert\sqrt{3}\right\rvert - \ln\left\lvert\dfrac{1}{\sqrt{3}}\right\rvert\right\} = 2\ln 3 \text{ or } \ln 9$	**A1** [3]

Q20

7	$\sqrt{x^2+y^2} = \dfrac{1}{1-\dfrac{x}{\sqrt{x^2+y^2}}} \Rightarrow \sqrt{x^2+y^2} - x = 1$	**M1A1**
	$\Rightarrow y^2 = 1 + 2x$ or equivalent RHS	**A1** [3]
	Sketch: Parabola symmetrical about x-axis. Intercepts at $\left(-\frac{1}{2}, 0\right)$ and $(0, \pm 1)$.	**B1 B1**
	Shading correct area	**B1** [3]
	Recognises $\dfrac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \dfrac{1}{(1-\cos\theta)^2}\,d\theta$ as the area of the region between parabola and y-axis.	
	$= 2\times \int_{-\frac{1}{2}}^{0} (1+2x)^{\frac{1}{2}}\,\text{d}x$	**M1**
	$= 2\left[\dfrac{1}{3}(1+2x)^{\frac{3}{2}}\right]_{-\frac{1}{2}}^{0} = \dfrac{2}{3}$	**M1A1** [4]

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Q21

5	$\sin\frac{1}{2}\theta=\frac{1}{2}\Rightarrow\theta=\frac{1}{3}\pi$; Intersection at $\left(\frac{1}{\sqrt{2}},\frac{1}{3}\pi\right)$ (Accept 1.05 for $\frac{1}{3}\pi$.)	M1A1 (2)
	C_1: Circle centre at pole and radius $1/\sqrt{2}$ C_2: Curve, approx. correct orientation, from (0,0) to $(1,\pi)$. Completely correct correct shape.	B1 B1 B1 (3)
	$\frac{1}{6}\times\pi\times\frac{1}{2}-\frac{1}{2}\int_0^{\frac{1}{3}\pi}\sin\frac{1}{2}\theta\,\mathrm{d}\theta$	M1A1
	$=\frac{1}{12}\pi\times\frac{1}{2}\left[-2\cos\frac{1}{2}\theta\right]_0^{\frac{1}{3}\pi}=\frac{1}{12}\pi+\frac{\sqrt{3}}{2}-1$	M1A1 (4) **Total: 9**

Q22

2	$s=\int_0^{\alpha}\sqrt{\mathrm{e}^{8\theta}+16\mathrm{e}^{8\theta}}\,\mathrm{d}\theta=\int_0^{\alpha}\sqrt{17}\mathrm{e}^{4\theta}\,\mathrm{d}\theta=$ (AEF) (LNR)	M1A1
	$=\left[\frac{\sqrt{17}}{4}\mathrm{e}^{4\theta}\right]_0^{\alpha}\Rightarrow\frac{\sqrt{17}}{4}\mathrm{e}^{4\alpha}-\frac{\sqrt{17}}{4}=2015$	dM1A1
	$\Rightarrow\mathrm{e}^{4\alpha}=1955.837\ \ldots\Rightarrow\alpha=1.89$ CAO	dM1A1 (6) **Total 6**

Q23

11O	Closed curve through pole with correct orientation. Completely correct.	**1** **1**]
	$2\times\frac{1}{2}a^2\int_{\frac{1}{2}\pi}^{\pi}(1-2\cos\theta+\cos^2\theta)\mathrm{d}\theta=a^2\int_{\frac{1}{2}\pi}^{\pi}\left(\frac{3}{2}-2\ \mathrm{s}\theta+\frac{1}{2}\ \mathrm{s}2\theta\right)\mathrm{d}\theta$	**1**
	$=a^2\left[\frac{3}{2}\theta-2\ \mathrm{n}\theta+\frac{1}{4}\ \mathrm{n}2\theta\right]_{\frac{1}{2}\pi}^{\pi}$	**1**
	$=a^2\left(\frac{3}{4}\pi+2\right)$	**1**]
	$\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2=a^2(1-2\ \mathrm{s}\theta+\ \mathrm{s}^2\theta+\ \mathrm{n}^2\theta)$	**1**
	$=2a^2(1-\ \mathrm{s}\theta)=2a^2.2\ \mathrm{n}^2\frac{1}{2}\theta=4a^2\ \mathrm{n}^2\frac{1}{2}\theta$ (AG)	**1**
	$=2\times\int_{\frac{1}{2}\pi}^{\pi}2a\ \mathrm{n}\frac{1}{2}\theta\,\mathrm{d}\theta$	**1**
	$=4a\left[-2\ \mathrm{s}\frac{1}{2}\theta\right]_{\frac{1}{2}\pi}^{\pi}$	**1**
	$=4\sqrt{2}a$	**1**] **1** **4**

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Q24

5	Closed curve through $(a, 0)$, $\left(2a, \frac{1}{2}\pi\right)$, (a, π), $\left(0, \frac{3}{2}\pi\right)$. Completely correct shape for cardioid.	B1 B1 [2]
	Area $= \frac{a^2}{2}\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}(1+\sin\theta)^2\,\mathrm{d}\theta = \frac{a^2}{2}\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}(1+2\sin\theta+\sin^2\theta)\mathrm{d}\theta$ (LNR)	M1
	$= \frac{a^2}{2}\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}\left(\frac{3}{2}+2\sin\theta-\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$ (LR)	M1
	$= \frac{a^2}{2}\left[\frac{3}{2}\theta-2\cos\theta-\frac{1}{4}\sin 2\theta\right]_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}$	M1
	$= a^2\left(\frac{1}{4}\pi+1+\frac{1}{8}\sqrt{3}\right)$ (CAO)	A1 [4]

Q25

4	Use of $r^2 = x^2 + y^2$ Use of $x = r\cos\theta$ and $y = r\sin\theta$ (both). Obtains $r^2 = a^2\sin 2\theta$ (AG)	B1 B1 B1 [3]
	Sketch with two loops, approximately symmetrical about $\theta = \frac{1}{4}\pi$ and $\theta = -\frac{3}{4}\pi$.	B1B1 [2]
	$\frac{1}{2}\int_0^{\frac{1}{2}\pi} a^2\sin 2\theta\,\mathrm{d}\theta = \left[-\frac{a^2}{4}\cos 2\theta\right]_0^{\frac{1}{2}\pi}$ (LNR)	M1
	$= \frac{1}{2}a^2$	A1 (2)

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Q26

8	Circle sketched Cardioid – correct location and orientation – correct indentation near pole.	B1 B1B1 (3)
	$\left(a, \frac{\pi}{2}\right)$ and $\left(a, \frac{3\pi}{2}\right)$ (B1 for reverse, or $a = a(1-\cos\theta) \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$ seen.)	B1B1 (2)
	Area $= \frac{1}{2}\pi a^2 + 2\times\frac{1}{2}\int_0^{\pi} a^2(1-\cos\theta)^2\,\mathrm{d}\theta$ (Half circle + Area of sector)	B1M1
	$= \frac{1}{2}\pi a^2 + a^2\int_0^{\pi}\left(1-2\cos\theta+\cos^2\theta\right)\mathrm{d}\theta$	A1
	$= \frac{1}{2}\pi a^2 + a^2\int_0^{\pi}\left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$ (Use of double angle formula.)	M1
	$= \frac{1}{2}\pi a^2 + a^2\left[\frac{3\theta}{2}-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$ (Integration)	M1
	$= \frac{1}{2}\pi a^2 + a^2\left(\frac{3}{4}\pi-2\right) = \left(\frac{5}{4}\pi-2\right)a^2$ (AG)	A1 (6) **[11]**

Q27

1	Use of $\frac{1}{2}\int r^2\,\mathrm{d}\theta$	$A = \frac{1}{2}\int_0^{2\pi} 4(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$	M1		
	Use of double angle formula and attempt to integrate.	$= \int_0^{2\pi}(3+4\cos\theta+\cos 2\theta)\,\mathrm{d}\theta$	M1		
	Integrates correctly.	$= \left[3\theta+4\sin\theta+\frac{\sin 2\theta}{2}\right]_0^{2\pi}$	A1		
	Finds value.	$= 6\pi$ (CWO) Accept 18.8	A1		**[4]**

Q28

10	Uses identity.	$2\sin\theta\cos\left(\theta-\frac{1}{4}\pi\right) = \sin\left(2\theta-\frac{1}{4}\pi\right)+\sin\left(\frac{1}{4}\pi\right)$	M1		
	Uses sine-cosine link.	$= \cos\left(\frac{1}{2}\pi-2\theta+\frac{1}{4}\pi\right)+\frac{1}{\sqrt{2}}$	M1		
	Obtains result.	$= \cos\left(2\theta-\frac{3}{4}\pi\right)+\frac{1}{\sqrt{2}}$ (AG)	A1	3	
	Sketches graph.	Closed loop through origin, in correct position.	B1		
	Obtains line of symmetry.	For line of symmetry $2\theta-\frac{3}{4}\pi=0 \Rightarrow \theta=\frac{3}{8}\pi$.	B1B1	3	
	Uses area of sector formula.	$A = \frac{1}{2}\int_0^{\frac{3}{4}\pi}\left\{\cos^2\left(2\theta-\frac{3}{4}\pi\right)+\sqrt{2}\cos\left(2\theta-\frac{3}{4}\pi\right)+\frac{1}{2}\right\}\mathrm{d}\theta$	M1		
	Rearranges.	$= \frac{1}{2}\int_0^{\frac{3}{4}\pi}\left\{\frac{1}{2}\cos\left[4\theta-\frac{3}{2}\pi\right]+\sqrt{2}\cos\left[2\theta-\frac{3}{4}\pi\right]+1\right\}\mathrm{d}\theta$	A1		
	Integrates correctly.	$= \left[\frac{1}{16}\sin\left(4\theta-\frac{3}{2}\pi\right)+\frac{1}{2\sqrt{2}}\sin\left(2\theta-\frac{3}{4}\pi\right)+\frac{\theta}{2}\right]_0^{\frac{3}{4}\pi}$	dM1A1		
	Substitutes limits.	$= \left[-\frac{1}{16}+\frac{1}{4}+\frac{3}{8}\pi\right]-\left[\frac{1}{16}-\frac{1}{4}\right]$	dM1		
	Obtains given answer.	$= \frac{3}{8}(\pi+1)$ (AG)	A1	6	**[12]**
		N.B Method marks are dependent in final par If $\frac{1}{2}$ factor missing throughout – award M's (Max 3) If $2\times\frac{1}{2}\int_0^{\frac{3}{8}} r^2\mathrm{d}\theta$, penultimate line is $= \left[\frac{3}{8}\pi\right]-\left[\frac{1}{8}-\frac{1}{2}\right]$			

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Q29

1 (i)	Uses area formula.	$\text{Area} = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\mathrm{e}^{2\theta}\mathrm{d}\theta = \left[\mathrm{e}^{2\theta}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$	M1		
	Obtains result.	$= \mathrm{e}^{\pi}-\mathrm{e}^{\frac{\pi}{3}}$ $(=20.3)$	A1	2	
(ii)	Uses arc length formula.	$\text{Arc length} = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sqrt{4\mathrm{e}^{2\theta}+4\mathrm{e}^{2\theta}}\,\mathrm{d}\theta = 2\sqrt{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\mathrm{e}^{\theta}\mathrm{d}\theta$	M1A1		
	Obtains result.	$= 2\sqrt{2}\left[\mathrm{e}^{\theta}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = 2\sqrt{2}\left[\mathrm{e}^{\frac{\pi}{2}}-\mathrm{e}^{\frac{\pi}{6}}\right]$ $(=8.83)$	A1	3	**[5]**

Q30

10	Differentiates wrt θ	$\frac{\mathrm{d}r}{\mathrm{d}\theta} = 2\cos\theta - 2\cos 2\theta$	M1		
	Equates to zero	$2c - 2\left(2c^2 - 1\right) = 0 \Rightarrow 2c^2 - c - 1 = 0$	A1		
	And solves equation	$\Rightarrow (2c+1)(c-1) = 0$	M1		
		$\Rightarrow c = -\frac{1}{2}$ or 1	A1		
	States required points on C	$\left(\frac{3}{2}\sqrt{3}, \frac{2}{3}\pi\right)$	A1	**(5)**	
	Sketches C.	Approximate shape and location Accurate scaling.	B1 B1	**(2)**	
	Uses $\frac{1}{2}\int r^2 \mathrm{d}\theta$	Area $= \frac{1}{2}\int_0^{\frac{\pi}{4}} 4\sin^2\theta\left(1 - 2\cos\theta + \cos^2\theta\right)\mathrm{d}\theta$	M1 M1		
	Obtains an integrable form	$= \int_0^{\frac{\pi}{4}}\left(1 - \cos 2\theta - 4\cos\theta\sin^2\theta + \frac{1}{4}[1 - \cos 4\theta]\right)\mathrm{d}\theta$	A1A1		
	Integrates	$= \left[\frac{5\theta}{4} - \frac{\sin 2\theta}{2} - 4\frac{\sin^3\theta}{3} - \frac{\sin 4\theta}{16}\right]_0^{\frac{\pi}{4}}$	M1		
	Obtains result	$= \frac{5}{16}\pi - \frac{1}{2} - \frac{\sqrt{2}}{3}$ or 0.0103	A1	**(6)**	**[13]**

Q31

4	Draws sketch of C.	Shows (4,0) and $(0,\pi)$ lie on C. Correct shape. (Full cardioid is B1 unless clear evidence of plotting up to 2π or $-\pi$ to π.)	B1 B1	2	
	Uses $\frac{1}{2}\int_\alpha^\beta r^2 \mathrm{d}\theta$	$\frac{1}{2}\int_0^\pi (4 + 8\cos\theta + 4\cos^2\theta)\mathrm{d}\theta$	M1		
	Uses double angle formula.	$= \int_0^\pi (2 + 4\cos\theta + 2\cos^2\theta)\mathrm{d}\theta$			
		$= \int_0^\pi (3 + 4\cos\theta + \cos 2\theta)\mathrm{d}\theta$	M1		
	Integrates and obtains area.	$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^\pi = 3\pi$ (A1 for **correct** integral)	A1A1 **CWO**	4	
	Finds areas.	$\frac{3\pi}{5} + 4\sin\frac{\pi}{5} + \sin\frac{\pi}{5}\cos\frac{\pi}{5} = 4.712$	M1A1		
		$3\pi - 4.712 = 4.713$	A1	3	**[9]**

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Q32

1	Re-writes equation	$r\cos\left(\theta-\frac{\pi}{4}\right)=\sqrt{2}$	M1		
	and uses compound angle formula.	$r\left(\frac{\cos\theta}{\sqrt{2}}+\frac{\sin\theta}{\sqrt{2}}\right)=\sqrt{2}$	A1		
	Changes to cartesian.	$\Rightarrow r\cos\theta+r\sin\theta=2\Rightarrow x+y=2$ or $y=2-x$.	A1	3	
	Sketches graph.	Straight line at $-\frac{\pi}{4}$ to the initial line.	B1		
		Point (2,0) clearly indicated.	B1	2	**[5]**

Q33

5	Sketches graph.	Correct shape and orientation. Passing through (0,0) and (3,0)	B1 B1	2	
	Uses area of sector formula.	Area $= 2\times\frac{1}{2}\int_0^{\frac{\pi}{3}}(1+2\cos\theta)^2\,\mathrm{d}\theta$	M1		
	Uses double angle formula Integrates.	$=\int_0^{\frac{\pi}{3}}(1+4\cos\theta+4\cos^2\theta)\mathrm{d}\theta$ $=\int_0^{\frac{\pi}{3}}(3+4\cos\theta+2\cos 2\theta)\,\mathrm{d}\theta$	M1		
	Obtains result.	$=\left[3\theta+4\sin\theta+\sin 2\theta\right]_0^{\frac{\pi}{3}}$ $=\left[\pi+\frac{5}{2}\sqrt{3}\right]$	A1 A1	4	**[6]**

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Q34

11	**EITHE**				
	Uses $x^2 + y^2 = r^2$, $x = r\cos\theta$ and $y = r\sin\theta$.	$(x^2 + y^2)^2 = a^2(x^2 - y^2)$	M1		
		$\Rightarrow r^2 = a^2\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right)$			
		$= a^2(\cos^2\theta - \sin^2\theta) = a^2\cos 2\theta$ (AG)	A1	2	
	One mark for each loop, or half of whole curve.	Sketches C.	B2,1,0	2	
	Uses sector area formula.	Area $= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a^2\cos 2\theta\,\mathrm{d}\theta \left(= \int_0^{\frac{\pi}{4}} a^2\cos 2\theta\,\mathrm{d}\theta\right)$	M1		
	S.C. Omission of factor, but correct integration gets B1.	$= a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = \frac{a^2}{2}$	A1A1	3	
	Differentiates.	$2(x^2 + y^2)(2x + 2yy') = a^2(2x - 2yy')$	B1B1		
	Puts $y' = 0$.	$y' = 0 \Rightarrow 2x(x^2 + y^2) = a^2x$	M1		
	Obtains coordinates.	$\Rightarrow 2r^2 = a^2 \Rightarrow r = \frac{a}{\sqrt{2}} \quad (r \geq 0)$	A1		
		$\Rightarrow \cos 2\theta = \frac{1}{2}$	M1		
		$\Rightarrow \theta = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$	A1A1	7	
		i. $\left(\frac{a}{\sqrt{2}}, \pm\frac{\pi}{6}\right)$ and $\left(\frac{a}{\sqrt{2}}, \pm\frac{5\pi}{6}\right)$			**[14]**
	Alternatively for last 7 marks:				
	Obtains condition for tangent parallel to initial line.	$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}\theta} = 0 \Rightarrow r\cos\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta = 0$	(M1A1)		
	Differentiates equation of C.	$2r\frac{\mathrm{d}r}{\mathrm{d}\theta} = -2a^2\sin 2\theta \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{a^2}{r}\sin 2\theta$	(A1)		
	Forms equation.	$\therefore r\cos\theta - \frac{a^2}{r}\sin 2\theta\sin\theta = 0$	(M1)		
		$\therefore a^2\cos 2\theta\cos\theta = a^2\sin 2\theta\sin\theta$			
		$\therefore \frac{1}{\tan 2\theta} = \tan\theta$			
		$\therefore \frac{1 - t^2}{2t} = t \Rightarrow 2t^2 = 1 - t^2 \Rightarrow 3t^2 = 1$	(M1)		
	Solves for tanθ,	$\therefore t = \pm\frac{1}{\sqrt{3}}$			
	and θ.	$\therefore \theta = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$	(A1)		
	Writes coordinates of points.	$\left(\frac{a}{\sqrt{2}}, \pm\frac{\pi}{6}\right)$, $\left(\frac{a}{\sqrt{2}}, \pm\frac{5\pi}{6}\right)$	(A1)	(7)	
		(Award final A1 if r found but result not written out.)			**[14]**

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VECTORS

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Q1

6 Let t be a positive constant.

The line l_1 passes through the point with position vector $t\mathbf{i}+\mathbf{j}$ and is parallel to the vector $-2\mathbf{i}-\mathbf{j}$. The line l_2 passes through the point with position vector $\mathbf{j}+t\mathbf{k}$ and is parallel to the vector $-2\mathbf{j}+\mathbf{k}$.

It is given that the shortest distance between the lines l_1 and l_2 is $\sqrt{21}$.

(a) Find the value of t. [5]

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The plane Π_1 contains l_1 and is parallel to l_2.

(b) Write down an equation of Π_1, giving your answer in the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$. [1]

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The plane Π_2 has Cartesian equation $5x - 6y + 7z = 0$.

(c) Find the acute angle between l_2 and Π_2. [3]

(d) Find the acute angle between Π_1 and Π_2. [3]

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Q2

6 The lines l_1 and l_2 have equations $\mathbf{r} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k} + s(2\mathbf{i} - 3\mathbf{j})$ and $\mathbf{r} = 3\mathbf{i} - 2\mathbf{k} + t(3\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ respectively.

The plane Π_1 contains l_1 and the point P with position vector $-2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

(a) Find an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. [2]

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The plane Π_2 contains l_2 and is parallel to l_1.

(b) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

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(c) Find the acute angle between Π_1 and Π_2. [5]

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(d) The point Q is such that $\overrightarrow{OQ} = -5\overrightarrow{OP}$.

Find the position vector of the foot of the perpendicular from the point Q to Π_2. [4]

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Q3

5 The plane Π has equation $\mathbf{r} = -2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + \mathbf{k}) + \mu(2\mathbf{i} + 3\mathbf{j})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

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The line l passes through the point P with position vector $2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and is parallel to the vector $\mathbf{k}$.

(b) Find the position vector of the point where l meets Π. [3]

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(c) Find the acute angle between l and Π. [3]

(d) Find the perpendicular distance from P to Π. [3]

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Q4

7 The points A, B, C have position vectors

$$2\mathbf{i}+2\mathbf{j}, \quad -\mathbf{j}+\mathbf{k} \quad \text{and} \quad 2\mathbf{i}+\mathbf{j}-7\mathbf{k}$$

respectively, relative to the origin O.

(a) Find an equation of the plane OAB, giving your answer in the form $\mathbf{r.n} = p$. [3]

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The plane Π has equation $x-3y-2z=1$.

(b) Find the perpendicular distance of Π from the origin. [1]

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(c) Find the acute angle between the planes OAB and Π. [3]

(d) Find an equation for the common perpendicular to the lines OC and AB. [10]

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Q5

4 The points A, B, C have position vectors

$$-\mathbf{i}+\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{i}-\mathbf{j}, \quad 2\mathbf{i}+2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

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(b) Find the perpendicular distance from O to the plane ABC. [2]

(c) Find the acute angle between the planes OAB and ABC. [4]

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Q6

7 The points A, B, C have position vectors

$$-2\mathbf{i}+2\mathbf{j}-\mathbf{k}, \quad -2\mathbf{i}+\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{j}+\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

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(b) Find the acute angle between the planes OBC and ABC. [4]

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The point D has position vector $t\mathbf{i} - \mathbf{j}$.

(c) Given that the shortest distance between the lines AB and CD is $\sqrt{10}$, find the value of t. [6]

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Q7

5 The lines l_1 and l_2 have equations $\mathbf{r} = 3\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} - 5\mathbf{j} - 6\mathbf{k} + \mu(5\mathbf{j} + 6\mathbf{k})$ respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

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The plane Π contains l_1 and is parallel to the vector $\mathbf{i}+\mathbf{k}$.

(b) Find the equation of Π, giving your answer in the form $ax+by+cz=d$. [4]

(c) Find the acute angle between l_2 and Π. [3]

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Q8

7 The lines l_1 and l_2 have equations $\mathbf{r} = -5\mathbf{j} + \lambda(5\mathbf{i} + 2\mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \mu(\mathbf{j} + \mathbf{k})$ respectively. The plane Π contains l_1 and is parallel to l_2.

(a) Find the equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

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(b) Find the distance between l_2 and Π. [3]

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The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(c) Show that P has position vector $\frac{55}{27}\mathbf{i} - 5\mathbf{j} + \frac{22}{27}\mathbf{k}$ and state a vector equation for PQ. [8]

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Q9

6 With O as the origin, the points A, B, C have position vectors

$$\mathbf{i} - \mathbf{j}, \quad 2\mathbf{i} + \mathbf{j} + 7\mathbf{k}, \quad \mathbf{i} - \mathbf{j} + \mathbf{k}$$

respectively.

(i) Find the shortest distance between the lines OC and AB. [5]

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(ii) Find the cartesian equation of the plane containing the line OC and the common perpendicular of the lines OC and AB. [4]

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Q10

3 The lines l_1 and l_2 have equations $\mathbf{r} = 6\mathbf{i} + 2\mathbf{j} + 7\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + \mu(-6\mathbf{j} + \mathbf{k})$ respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vectors of P and Q. [8]

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Q11

7 The line l_1 passes through the points $A\,(-3,\ 1,\ 4)$ and $B\,(-1,\ 5,\ 9)$. The line l_2 passes through the points $C\,(-2,\ 6,\ 5)$ and $D\,(-1,\ 7,\ 5)$.

(i) Find the shortest distance between the lines l_1 and l_2. [5]

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(ii) Find the acute angle between the line l_2 and the plane containing A, B and D. [5]

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Q12

10 The line l_1 is parallel to the vector $a\mathbf{i} - \mathbf{j} + \mathbf{k}$, where a is a constant, and passes through the point whose position vector is $9\mathbf{j} + 2\mathbf{k}$. The line l_2 is parallel to the vector $-a\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and passes through the point whose position vector is $-6\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$.

(i) It is given that l_1 and l_2 intersect.

(a) Show that $a = -\frac{6}{13}$. [3]

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(b) Find a cartesian equation of the plane containing l_1 and l_2. [4]

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(ii) Given instead that the perpendicular distance between l_1 and l_2 is $3\sqrt{(30)}$, find the value of a. [5]

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Q13

7 The lines l_1 and l_2 have vector equations

$$\mathbf{r} = a\mathbf{i} + 9\mathbf{j} + 13\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \quad \text{and} \quad \mathbf{r} = -3\mathbf{i} + 7\mathbf{j} - 2\mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$$

respectively. It is given that l_1 and l_2 intersect.

(i) Find the value of the constant a. [3]

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The point P has position vector $3\mathbf{i} + \mathbf{j} + 6\mathbf{k}$.

(ii) Find the perpendicular distance from P to the plane containing l_1 and l_2. [4]

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(iii) Find the perpendicular distance from P to l_2. [4]

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Q14

8 The plane Π_1 has equation

$$\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

(i) Find a cartesian equation of Π_1. [3]

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The plane Π_2 has equation $3x + y - z = 3$.

(ii) Find the acute angle between Π_1 and Π_2, giving your answer in degrees. [2]

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(iii) Find an equation of the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. [5]

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Q15

10 The position vectors of the points A, B, C, D are

$$\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \qquad 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}, \qquad -\mathbf{i} + 3\mathbf{k}, \qquad m\mathbf{j} + 4\mathbf{k},$$

respectively, where m is a constant.

(i) Show that the lines AB and CD are parallel when $m = \frac{3}{2}$. [1]

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(ii) Given that $m \neq \frac{3}{2}$, find the shortest distance between the lines AB and CD. [5]

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(iii) When $m = 2$, find the acute angle between the planes ABC and ABD, giving your answer in degrees. [6]

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Q16

OR

The position vectors of the points A, B, C, D are

$$\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}, \quad 3\mathbf{i} - \mathbf{j} + \mathbf{k}, \quad 5\mathbf{i} - 5\mathbf{j} + \alpha\mathbf{k},$$

respectively, where α is a positive integer. It is given that the shortest distance between the line AB and the line CD is equal to $2\sqrt{2}$.

(i) Show that the possible values of α are 3 and 5. [7]

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(ii) Using $\alpha = 3$, find the shortest distance of the point D from the line AC, giving your answer correct to 3 significant figures. [3]

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(iii) Using $\alpha = 3$, find the acute angle between the planes ABC and ABD, giving your answer in degrees. [4]

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Q17

9 The plane Π_1 passes through the points $(1, 2, 1)$ and $(5, -2, 9)$ and is parallel to the vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

(i) Find the cartesian equation of Π_1. [4]

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The plane Π_2 contains the lines

$$\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \quad \text{and} \quad \mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}).$$

(ii) Find the cartesian equation of Π_2. [4]

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(iii) Find the acute angle between Π_1 and Π_2. [3]

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Q18

6 The points A, B and C have position vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ respectively.

(i) Find the area of the triangle ABC. [4]

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(ii) Find the perpendicular distance of the point A from the line BC. [3]

(iii) Find the cartesian equation of the plane through A, B and C. [2]

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8 Find a cartesian equation of the plane Π_1 passing through the points with coordinates (2, −1, 3), (4, 2, −5) and (−1, 3, −2). [4]

The plane Π_2 has cartesian equation $3x - y + 2z = 5$. Find the acute angle between Π_1 and Π_2. [3]

Find a vector equation of the line of intersection of the planes Π_1 and Π_2. [4]

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Q20

11: The position vectors of the points A, B, C, D are

$$\mathbf{a} = 2\mathbf{i} + \lambda\mathbf{j} - 3\mathbf{k}, \quad \mathbf{b} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}, \quad \mathbf{c} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{d} = \mathbf{i} + 7\mathbf{j} + 4\mathbf{k}$$

respectively. It is given that the shortest distance between the lines AB and CD is 3.

(i) Show that $\lambda^2 + \lambda - 20 = 0$. [7]

(ii) The planes p_1 and p_2 are the planes through A, B and D corresponding to the two values of λ satisfying the equation in part **(i)**. Find the acute angle between p_1 and p_2. [7]

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[Turn over

11: **EITHER**

The lines l_1 and l_2 have equations

$$\mathbf{r} = 6\mathbf{i} - 3\mathbf{j} + s(3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}) \quad \text{and} \quad \mathbf{r} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k} + t(\mathbf{i} - 3\mathbf{j} - \mathbf{k})$$

respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Show that the position vector of P is $3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and find the position vector of Q. [7]

Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, an equation of the plane Π which passes through P and is perpendicular to l_1. [3]

The plane Π meets the plane $\mathbf{r} = p\mathbf{i} + q\mathbf{j}$ in the line l_3. Find a vector equation of l_3. [4]

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Q22

11: The lines l_1 and l_2 have equations $\mathbf{r} = 8\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j})$ and $\mathbf{r} = 5\mathbf{i} + 3\mathbf{j} - 14\mathbf{k} + \mu(2\mathbf{j} - 3\mathbf{k})$ respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vector of the point P and the position vector of the point Q. [8]

The points with position vectors $8\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $5\mathbf{i} + 3\mathbf{j} - 14\mathbf{k}$ are denoted by A and B respectively. Find

(i) $\overrightarrow{AP} \times \overrightarrow{AQ}$ and hence the area of the triangle APQ,

(ii) the volume of the tetrahedron $APQB$. (You are given that the volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height.)

[6]

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[Turn over

Q23

8 A line, passing through the point $A\,(3,\ 0,\ 2)$, has vector equation $\mathbf{r} = 3\mathbf{i} + 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$. It meets the plane Π, which has equation $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 3$, at the point P. Find the coordinates of P. [3]

Write down a vector $\mathbf{n}$ which is perpendicular to Π, and calculate the vector $\mathbf{w}$, where

$$\mathbf{w} = \mathbf{n} \times (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$ [3]

The point Q lies in Π and is the foot of the perpendicular from A to Π. Use the vector $\mathbf{w}$ to determine an equation of the line PQ in the form $\mathbf{r} = \mathbf{u} + \mu\mathbf{v}$. [4]

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[Turn over

Q24

11 **EITHER**

The points A, B and C have position vectors $\mathbf{i}$, $2\mathbf{j}$ and $4\mathbf{k}$ respectively, relative to an origin O. The point N is the foot of the perpendicular from O to the plane ABC. The point P on the line-segment ON is such that $OP = \frac{3}{4}ON$. The line AP meets the plane OBC at Q. Find a vector perpendicular to the plane ABC and show that the length of ON is $\dfrac{4}{\sqrt{(21)}}$. [4]

Find the position vector of the point Q. [5]

Show that the acute angle between the planes ABC and ABQ is $\cos^{-1}\left(\frac{2}{3}\right)$. [5]

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[Turn over

11 The line l_1 passes through the points $A\,(2,\ 3,\ -5)$ and $B\,(8,\ 7,\ -13)$. The line l_2 passes through the points $C\,(-2,\ 1,\ 8)$ and $D\,(3,\ -1,\ 4)$. Find the shortest distance between the lines l_1 and l_2. [5]

The plane Π_1 passes through the points A, B and D. The plane Π_2 passes though the points A, C and D. Find the acute angle between Π_1 and Π_2, giving your answer in degrees. [6]

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Q26
11: **OR**

With respect to an origin O, the point A has position vector $4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and the plane Π_1 has equation

$$\mathbf{r} = (4 + \lambda + 3\mu)\mathbf{i} + (-2 + 7\lambda + \mu)\mathbf{j} + (2 + \lambda - \mu)\mathbf{k},$$

where λ and μ are real. The point L is such that $\overrightarrow{OL} = 3\overrightarrow{OA}$ and Π_2 is the plane through L which is parallel to Π_1. The point M is such that $\overrightarrow{AM} = 3\overrightarrow{ML}$.

(i) Show that A is in Π_1. [1]

(ii) Find a vector perpendicular to Π_2. [2]

(iii) Find the position vector of the point N in Π_2 such that ON is perpendicular to Π_2. [5]

(iv) Show that the position vector of M is $10\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$ and find the perpendicular distance of M from the line through O and N, giving your answer correct to 3 significant figures. [6]

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10 The line l_1 is parallel to the vector $\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ and passes through the point A, whose position vector is $3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$. The line l_2 is parallel to the vector $-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and passes through the point B, whose position vector is $-3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find

(i) the length PQ, [5]

(ii) the cartesian equation of the plane Π containing PQ and l_2, [4]

(iii) the perpendicular distance of A from Π. [3]

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Q28

11: **OR**

The points A, B, C and D have coordinates as follows:

$$A\,(2,\ 1,\ -2), \quad B\,(4,\ 1,\ -1), \quad C\,(3,\ -2,\ -1) \quad \text{and} \quad D\,(3,\ 6,\ 2).$$

The plane Π_1 passes through the points A, B and C. Find a cartesian equation of Π_1. [4]

Find the area of triangle ABC and hence, or otherwise, find the volume of the tetrahedron $ABCD$. [6]

[The volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height.]

The plane Π_2 passes through the points A, B and D. Find the acute angle between Π_1 and Π_2. [4]

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Q29

11 **EITHER**

The line l_1 passes through the point A whose position vector is $4\mathbf{i} + 7\mathbf{j} - \mathbf{k}$ and is parallel to the vector $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The line l_2 passes through the point B whose position vector is $\mathbf{i} + 7\mathbf{j} + 11\mathbf{k}$ and is parallel to the vector $\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$. The points P on l_1 and Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vectors of P and Q. [8]

Find the shortest distance between the line through A and B and the line through P and Q, giving your answer correct to 3 significant figures. [6]

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Q30

8 The plane Π_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$. Find a cartesian equation of Π_1. [3]

The plane Π_2 has equation $2x - y + z = 10$. Find the acute angle between Π_1 and Π_2. [2]

Find an equation of the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. [5]

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[Turn over

8 The points A, B, C have position vectors

$$4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}, \qquad 5\mathbf{i} + 7\mathbf{j} + 8\mathbf{k}, \qquad 2\mathbf{i} + 6\mathbf{j} + 4\mathbf{k},$$

respectively, relative to the origin O. Find a cartesian equation of the plane ABC. [4]

The point D has position vector $6\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$. Find the coordinates of E, the point of intersection of the line OD with the plane ABC. [4]

Find the acute angle between the line ED and the plane ABC. [3]

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11 **OR**

The position vectors of the points A, B, C, D are

$$2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}, \quad -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}, \quad \mathbf{i} + 4\mathbf{j} + \mathbf{k}, \quad \mathbf{i} + 5\mathbf{j} + m\mathbf{k},$$

respectively, where m is an integer. It is given that the shortest distance between the line through A and B and the line through C and D is 3. Show that the only possible value of m is 2. [7]

Find the shortest distance of D from the line through A and C. [3]

Show that the acute angle between the planes ACD and BCD is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$. [4]

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9 The plane Π has equation

$$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) + \mu(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$

The line l, which does not lie in Π, has equation

$$\mathbf{r} = 3\mathbf{i} + 6\mathbf{j} + 12\mathbf{k} + t(8\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}).$$

Show that l is parallel to Π. [4]

Find the position vector of the point at which the line with equation $\mathbf{r} = 5\mathbf{i} - 4\mathbf{j} + 7\mathbf{k} + s(2\mathbf{i} - \mathbf{j} + \mathbf{k})$ meets Π. [4]

Find the perpendicular distance from the point with position vector $9\mathbf{i} + 11\mathbf{j} + 2\mathbf{k}$ to l. [4]

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4 The points A, B and C have position vectors $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ respectively. Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [3]

Deduce, in either order, the exact value of

(i) the area of the triangle ABC,

(ii) the perpendicular distance from C to AB.

[3]

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Q35

9 The plane Π_1 has parametric equation

$$\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}).$$

Find a cartesian equation of Π_1. [4]

The plane Π_2 has cartesian equation $3x - 2y - 3z = 4$. Find the acute angle between Π_1 and Π_2. [3]

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

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VECTORS
MS

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Q1

Question	Answer	Marks	Guidance
6(a)	$\begin{pmatrix}0\\1\\t\end{pmatrix}-\begin{pmatrix}t\\1\\0\end{pmatrix}=t\begin{pmatrix}-1\\0\\1\end{pmatrix}$	**B1**	Finds vector from any point on l_1 to any point on l_2.
	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-2&-1&0\\0&-2&1\end{vmatrix}=\begin{pmatrix}-1\\2\\4\end{pmatrix}$	**M1 A1**	Finds common perpendicular.
	$\frac{t}{\sqrt{21}}\left\|\begin{pmatrix}-1\\0\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\4\end{pmatrix}\right\|=\sqrt{21}\Rightarrow t=\frac{21}{5}=4.2$	**M1 A1**	Uses formula for shortest distance.
		5	
6(b)	$\mathbf{r}=\frac{21}{5}\mathbf{i}+\mathbf{j}+\lambda(2\mathbf{i}+\mathbf{j})+\mu(-2\mathbf{j}+\mathbf{k})$	**B1 FT**	Using their value of t.
		1	
6(c)	$\begin{pmatrix}0\\-2\\1\end{pmatrix}\bullet\begin{pmatrix}5\\-6\\7\end{pmatrix}=\sqrt{5}\sqrt{110}\cos\alpha$ leading to $\cos\alpha=\frac{19}{\sqrt{5}\sqrt{110}}$	**M1 A1**	Uses dot product of $-2\mathbf{j}+\mathbf{k}$ and normal.
	Acute angle between l_2 and Π_2 is $90-\alpha=54.1^\circ$	**A1**	
		3	
6(d)	$\begin{pmatrix}-1\\2\\4\end{pmatrix}\bullet\begin{pmatrix}5\\-6\\7\end{pmatrix}=\sqrt{21}\sqrt{110}\cos\alpha$ leading to $\cos\alpha=\frac{11}{\sqrt{21}\sqrt{110}}$	**M1 A1 FT**	Dot product using their normal to Π_1.
	76.8°	**A1**	
		3	

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Q2

Question	Answer	Marks	Guidance
6(a)	$-(-2\mathbf{i}-2\mathbf{j}+4\mathbf{k})+(-\mathbf{i}-2\mathbf{j}+\mathbf{k})=\mathbf{i}-3\mathbf{k}$	**B1**	OE. Finds direction vector from P to a point of l_1.
	$\mathbf{r}=-2\mathbf{i}-2\mathbf{j}+4\mathbf{k}+\lambda(2\mathbf{i}-3\mathbf{j})+\mu(\mathbf{i}-3\mathbf{k})$ $\boldsymbol{r}=-\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}+\lambda(2i-3j)+\mu(\boldsymbol{i}-3\boldsymbol{k})$	**B1 FT**	OE. FT *their* $\mathbf{i}-3\mathbf{k}$
		2	
6(b)	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-3&0\\3&-1&3\end{vmatrix}=\begin{pmatrix}-9\\-6\\7\end{pmatrix}$	**M1 A1**	OE. Finds vector perpendicular to Π_2.
	$-9(3)-6(0)+7(-2)=-41$	**M1**	Uses point on Π_2.
	$9x+6y-7z=41$	**A1**	
		4	
6(c)	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-3&0\\1&0&-3\end{vmatrix}=\begin{pmatrix}9\\6\\3\end{pmatrix}\sim\begin{pmatrix}3\\2\\1\end{pmatrix}$	**M1 A1**	Finds vector perpendicular to Π_1.
	$\begin{pmatrix}3\\2\\1\end{pmatrix}\bullet\begin{pmatrix}9\\6\\-7\end{pmatrix}=\sqrt{14}\sqrt{166}\cos\alpha$ leading to $\cos\alpha=\dfrac{32}{\sqrt{14}\sqrt{166}}$	**DM1** **A1 FT**	Dot product using their normal vectors.
	48.4°	**A1**	0.845 rad
		5	

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6(d)	$\overrightarrow{OF} = \overrightarrow{OQ} + \overrightarrow{QF} = -5\begin{pmatrix}-2\\-2\\4\end{pmatrix} + t\begin{pmatrix}9\\6\\-7\end{pmatrix} = \begin{pmatrix}10+9t\\10+6t\\-20-7t\end{pmatrix}$	**M1 A1 FT**	Scales OP by a factor of ± 5 and uses multiple of their normal to Π_2.
	$9(10+9t)+6(10+6t)-7(-20-7t)=41$ leading to $290+166t=41$	**DM1**	Substitutes into the equation of Π_2.
	$t=-\frac{3}{2}$ leading to $\overrightarrow{OF} = \begin{pmatrix}-\frac{7}{2}\\1\\-\frac{19}{2}\end{pmatrix}$	**A1**	
		4	

Q3

Question	Answer	Marks	Guidance
5(a)	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\1 & 0 & 1\\2 & 3 & 0\end{vmatrix} = \begin{pmatrix}-3\\2\\3\end{pmatrix}$	**M1 A1**	Finds vector perpendicular to Π.
	$-3(-2)+2(3)+3(3)=21$	**M1**	Substitutes point on Π.
	$-3x+2y+3z=21$	**A1**	
		4	

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5(b)	$\begin{pmatrix} 2 \\ -3 \\ 5+t \end{pmatrix}$	**B1**	Forms general point on line (given as a single vector).
	$-3(2)+2(-3)+3(5+t)=21$ leading to $t=6$	**M1**	Substitutes into the equation for Π.
	$\begin{pmatrix} 2 \\ -3 \\ 11 \end{pmatrix}$	**A1**	
		3	
5(c)	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix} = \sqrt{1}\sqrt{22}\cos\alpha$ leading to $\cos\alpha = \frac{3}{\sqrt{22}}$	**M1** **A1 FT**	Uses dot product of **k** and their normal.
	Acute angle between l and Π is $90-\alpha = 39.8^\circ$	**A1**	
		3	
5(d)	$\begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$	**B1**	Finds direction vector from P to plane.
	$\frac{1}{\sqrt{22}}\begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix} = \frac{18}{\sqrt{22}} = 3.84$	**M1 A1**	Uses dot product of their direction and normal vectors.
		3	

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Q4

Question	Answer	Marks	Guidance
7(a)	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$	**M1 A1**	Finds common perpendicular.
	$\mathbf{r}.\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 0$	**A1**	
		3	
7(b)	$\frac{1}{\sqrt{1^2+3^2+2^2}} = \frac{1}{\sqrt{14}}$	**B1**	Divides by magnitude of the normal to Π 0.267
		1	
7(c)	$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} = \sqrt{3}\sqrt{14}\cos\alpha$ leading to $\cos\alpha = \frac{6}{\sqrt{3}\sqrt{14}}$	**M1** **A1 FT**	Takes dot product of normal vectors.
	22.2°	**A1**	Accept 0.388 radians. Mark final answer.
		3	

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7(d)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -7 \\ -2 & -3 & 1 \end{vmatrix} = \begin{pmatrix} -20 \\ 12 \\ -4 \end{pmatrix} \sim \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$	**M1 A1**	Finds direction of common perpendicular.
	$\overrightarrow{OP} = \lambda \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix}, \overrightarrow{OQ} = \begin{pmatrix} 2-2\mu \\ 2-3\mu \\ \mu \end{pmatrix}$ leading to $\overrightarrow{PQ} = \begin{pmatrix} 2-2\mu-2\lambda \\ 2-3\mu-\lambda \\ \mu+7\lambda \end{pmatrix}$	**M1 A1**	Finds $\overrightarrow{PQ}$.
	$\begin{pmatrix} 2-2\mu-2\lambda \\ 2-3\mu-\lambda \\ \mu+7\lambda \end{pmatrix} \bullet \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} = 0$ or $\begin{pmatrix} 2-2\mu-2\lambda \\ 2-3\mu-\lambda \\ \mu+7\lambda \end{pmatrix} = k \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$	**M1**	Uses that dot product of $\overrightarrow{PQ}$ with line direction is zero, or, alternatively, $\overrightarrow{PQ}$ is a multiple of the common perpendicular (parameter k not 1).
		A1	Deduces one equation.
	$\begin{pmatrix} 2-2\mu-2\lambda \\ 2-3\mu-\lambda \\ \mu+7\lambda \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix} = 0 \Rightarrow 14\mu + 54\lambda = 6$	**A1**	Deduces second equation.
	$\lambda = -\frac{1}{10}$ leading to $\overrightarrow{OP} = -\frac{1}{10} \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix}$	**M1 A1**	Solves for λ and substitutes into $\overrightarrow{OP}$.
	$\mathbf{r} = \begin{pmatrix} -0.2 \\ -0.1 \\ 0.7 \end{pmatrix} + k \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$	**B1 FT**	FT using their common perpendicular.
		10	

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Q5

Question	Answer	Marks	Guidance
4(a)	$\overrightarrow{AB} = -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} \quad \overrightarrow{AC} = 3\mathbf{i} - \mathbf{j}$	**B1**	Finds direction vectors of two lines in the plane. $\overrightarrow{BC} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 3 & -1 & 0 \end{vmatrix} = \begin{pmatrix} -2 \\ -6 \\ 7 \end{pmatrix}$	**M1 A1**	Finds normal to the plane ABC.
	$-2(-1) - 6(1) + 7(2) = 10 \Rightarrow -2x - 6y + 7z = 10$	**M1 A1**	Substitutes point.
	Alternative method for question 4(a)		
	Setting up 3 equations using points given.	**M1**	
	$-2x - 6y + 7z = 10$	**A1 A1 A1 A1**	OE
		5	
4(b)	$\frac{10}{\sqrt{2^2 + 6^2 + \ ^2}} = \frac{10}{\sqrt{89}}$ OE	**M1 A1**	Divides by magnitude of normal vector. 1.06…
		2	
4(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ -2 & -1 & 0 \end{vmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$	**M1 A1**	Finds normal to the plane OAB.
	$\begin{pmatrix} -2 \\ -6 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} = \sqrt{89}\sqrt{29}\cos\theta$	**M1**	Uses dot product correctly.
	36.2°	**A1**	
		4	

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Q6

Question	Answer	Marks	Guidance
7(a)	$\overrightarrow{AB} = -\mathbf{j} + 3\mathbf{k} \quad \overrightarrow{AC} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$	**B1**	Finds direction vectors of two lines in the plane. $\overrightarrow{BC} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 3 \\ 1 & -2 & 1 \end{vmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$	**M1 A1**	Finds normal to the plane ABC. OE
	$5(0) + 3(-2) + 1(1) = -5 \Rightarrow 5x + 3y + z = -5$	**M1 A1**	Substitutes point. OE
		5	
7(b)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$	**M1 A1**	Finds normal to the plane OBC.
	$\begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} = \sqrt{35}\sqrt{45}\cos\theta$	**M1**	Uses dot product of normal vectors.
	$28.1°$	**A1**	0.49(0) rad
7(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 3 \\ t & 1 & -1 \end{vmatrix} = \begin{pmatrix} -2 \\ 3t \\ t \end{pmatrix}$	**M1 A1** **4**	Finds common perpendicular.
	$\left\lvert \frac{1}{\sqrt{4+10t^2}} \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 3t \\ t \end{pmatrix} \right\rvert = \left\lvert \frac{-4-10t}{\sqrt{4+10t^2}} \right\rvert$	**M1 A1**	Uses formula for perpendicular distance.
	$\left\lvert \frac{-4-10t}{\sqrt{4+10t^2}} \right\rvert = \sqrt{10} \Rightarrow (4+10t)^2 = 10\left(4+10t^2\right)$	**M1**	Sets equal to $\sqrt{10}$ and solves for t.
	$t = \frac{3}{10}$	**A1**	
		6	

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Q7

Question	Answer	Marks
5(a)	$\begin{pmatrix}3\\-5\\-6\end{pmatrix}-\begin{pmatrix}3\\0\\3\end{pmatrix}=\begin{pmatrix}0\\-5\\-9\end{pmatrix}$	**B1**
	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&4&4\\0&5&6\end{vmatrix}=\begin{pmatrix}4\\-6\\5\end{pmatrix}$	**M1 A1**
	$\frac{1}{\sqrt{77}}\left\|\begin{pmatrix}0\\-5\\-9\end{pmatrix}\bullet\begin{pmatrix}4\\-6\\5\end{pmatrix}\right\|=\frac{15}{\sqrt{77}}=1.71$	**M1 A1**
		5
5(b)	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&4&4\\1&0&1\end{vmatrix}=\begin{pmatrix}4\\3\\-4\end{pmatrix}$	**M1 A1**
	$4(3)+3(0)-4(3)=0\Rightarrow 4x+3y-4z=0$	**M1 A1**
		4
5(c)	$\begin{pmatrix}0\\5\\6\end{pmatrix}\bullet\begin{pmatrix}4\\3\\-4\end{pmatrix}=\sqrt{61}\sqrt{41}\cos\alpha\Rightarrow\cos\alpha=\frac{-9}{\sqrt{61}\sqrt{41}}$ (**A1 FT** *their* normal)	**M1 A1FT**
	Acute angle between l_2 and Π is $\alpha-90=10.4^\circ$	**A1**
		3

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Q8

Question	Answer	Marks
7(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix}$	**M1 A1**
	$-5(-5) = 25$	**M1**
	$-2x - 5y + 5z = 25$	**A1**
		4
7(b)	$\begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -7 \\ 2 \end{pmatrix}$	**M1**
	$\frac{1}{\sqrt{54}} \left\| \begin{pmatrix} -4 \\ -7 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix} \right\| = \frac{53}{\sqrt{54}} = 7.21$	**M1 A1**
		3

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7(c)	$\overrightarrow{OP}=\begin{pmatrix}5\lambda\\-5\\2\lambda\end{pmatrix}, \overrightarrow{OQ}=\begin{pmatrix}4\\2+\mu\\-2+\mu\end{pmatrix}\Rightarrow \overrightarrow{PQ}=\begin{pmatrix}4-5\lambda\\7+\mu\\-2+\mu-2\lambda\end{pmatrix}$	**M1 A1**
	$\begin{pmatrix}4-5\lambda\\7+\mu\\-2+\mu-2\lambda\end{pmatrix}\bullet\begin{pmatrix}5\\0\\2\end{pmatrix}=0$ or $\begin{pmatrix}4-5\lambda\\7+\mu\\-2+\mu-2\lambda\end{pmatrix}=k\begin{pmatrix}-2\\-5\\5\end{pmatrix}$	**M1**
		A1
	$\begin{pmatrix}4-5\lambda\\7+\mu\\-2+\mu-2\lambda\end{pmatrix}\bullet\begin{pmatrix}0\\1\\1\end{pmatrix}=0\Rightarrow -2\lambda+2\mu=-5$	**A1**
	$\lambda=\frac{11}{27}\Rightarrow\overrightarrow{OP}=\frac{1}{27}\begin{pmatrix}55\\-135\\22\end{pmatrix}=\frac{55}{27}\mathbf{i}-5\mathbf{j}+\frac{22}{27}\mathbf{k}$	**M1 A1**
	$\mathbf{r}=\frac{1}{27}\begin{pmatrix}55\\-135\\22\end{pmatrix}+k\begin{pmatrix}-2\\-5\\5\end{pmatrix}$ (**B1 FT** their common perpendicular)	**B1 FT**
		8

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Q9

Question	Answer	Marks	Guidance
6(i)	$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$	**B1**	
	$\overrightarrow{OC} \times \overrightarrow{AB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 2 & 7 \end{vmatrix} = \begin{pmatrix} -9 \\ -6 \\ 3 \end{pmatrix} = t \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$	**M1 A1**	Finds direction of common perpendicular.
	$\frac{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}}{\sqrt{3^2 + 2^2 + 1^2}} = \frac{1}{\sqrt{14}} = 0.267$	**M1 A1**	Uses formula for shortest distance.
		5	
6(ii)	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 3 & 2 & -1 \end{vmatrix} = t \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$	**M1 A1**	Finds normal to plane.
	$-(0) + 4(0) + 5(0) = 0$	**M1**	Uses point on plane.
	$-x + 4y + 5z = 0$	**A1**	AEF
		4	

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Q10

Question	Answer	Marks	Guidance
3	$\overrightarrow{OP}=\begin{pmatrix}6+\lambda\\2+\lambda\\7\end{pmatrix},\overrightarrow{OQ}=\begin{pmatrix}4\\4-6\mu\\\mu\end{pmatrix}\Rightarrow\overrightarrow{PQ}=\begin{pmatrix}-2-\lambda\\2-\lambda-6\mu\\-7+\mu\end{pmatrix}$	**M1 A1**	Finds $\overrightarrow{PQ}$.
	$\begin{pmatrix}-2-\lambda\\2-\lambda-6\mu\\-7+\mu\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}=0$ **Or** $\begin{pmatrix}-2-\lambda\\2-\lambda-6\mu\\-7+\mu\end{pmatrix}=k\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\0&-6&1\end{vmatrix}=k\begin{pmatrix}1\\-1\\-6\end{pmatrix}$	**M1**	Uses that dot product of $\overrightarrow{PQ}$ with line directions is 0. Or, alternatively, $\overrightarrow{PQ}$ is multiple of common perpendicular.
	$-2\lambda-6\mu=0$	**A1**	Deduces one equation. CWO.
	$\begin{pmatrix}-2-\lambda\\2-\lambda-6\mu\\-7+\mu\end{pmatrix}\begin{pmatrix}0\\-6\\1\end{pmatrix}=0\Rightarrow6\lambda+37\mu=19$	**A1**	Deduces second equation. CWO.
	$\lambda=-3,\ \mu=1$	**M1 A1**	Solves simultaneous equations.
	$\overrightarrow{OP}=\begin{pmatrix}3\\-1\\7\end{pmatrix},\overrightarrow{OQ}=\begin{pmatrix}4\\-2\\1\end{pmatrix}$	**A1**	States $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.
		8	

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Q11

Question	Answer	Marks	Guidance
7(i)	$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}, \overrightarrow{CD} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$	**B1**	Find the directions of the lines.
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 5 \\ 1 & 1 & 0 \end{vmatrix} = \begin{pmatrix} -5 \\ 5 \\ -2 \end{pmatrix}$	**M1 A1**	Finds direction of common perpendicular. Allow any non-zero scalar multiple.
	$\dfrac{\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 5 \\ -2 \end{pmatrix}}{\sqrt{5^2 + 5^2 + 2^2}} = \dfrac{18}{\sqrt{54}} = \sqrt{6} = 2.45$	**M1 A1**	Uses formula for shortest distance.
		5	
7(ii)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 5 \\ 2 & 6 & 1 \end{vmatrix} = \begin{pmatrix} -26 \\ 8 \\ 4 \end{pmatrix} \sim \begin{pmatrix} -13 \\ 4 \\ 2 \end{pmatrix}$	**M1 A1**	Finds normal to the plane.
	$\dfrac{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -13 \\ 4 \\ 2 \end{pmatrix}}{\sqrt{1^2 + 1^2 + 0^2}\sqrt{13^2 + 4^2 + 2}} = -\dfrac{9}{\sqrt{378}} = -\dfrac{\sqrt{42}}{14}$	**M1 A1**	**FT** Uses dot product of their normal with direction of line.
	$\cos^{-1}\left(-\dfrac{\sqrt{42}}{14}\right) - 90 = 27.6° \quad (\text{or } 0.481 \text{ rad})$	**A1**	
		5	

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Q12

Question	Answer	Marks	Guidance
10(i)(a)	Point on l_1 is $\begin{pmatrix} \lambda a \\ 9-\lambda \\ 2+\lambda \end{pmatrix}$ and on l_2 is $\begin{pmatrix} -6-\mu a \\ -5+2\mu \\ 10+4\mu \end{pmatrix}$	**B1**	
	Point of intersection: $\begin{pmatrix} \lambda a \\ 9-\lambda \\ 2+\lambda \end{pmatrix} = \begin{pmatrix} -6-\mu a \\ -5+2\mu \\ 10+4\mu \end{pmatrix}$	**M1**	Equates coordinates of points
	$\Rightarrow \mu = 1,\ \lambda = 12,\ a = -\frac{6}{13}$	**A1**	AG
		3	
10(i)(b)	Normal to the plane: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{6}{13} & -1 & 1 \\ \frac{6}{13} & 2 & 4 \end{vmatrix} = -6\mathbf{i} + \frac{30}{13}\mathbf{j} - \frac{6}{13}\mathbf{k} \sim -13\mathbf{i} + 5\mathbf{j} - \mathbf{k}$	**M1 A1**	Uses cross product to find normal to the plane AEF
	Using point on plane: e.g. $5(9) - 2 = 43$	**M1**	Substitutes a point
	Equation of plane: $-13x + 5y - z = 43$	**A1**	AEF
		4	

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10(ii)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & -1 & 1 \\ -a & 2 & 4 \end{vmatrix} = -6i - 5aj + ak$ $\left\lvert \begin{pmatrix} -6 \\ -5a \\ a \end{pmatrix} \right\rvert = \sqrt{36 + 26a^2}$	**B1**	Finds the magnitude of the cross product of the direction vectors of the lines
	$\begin{pmatrix} -6 \\ -5-9 \\ 10-2 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ -5a \\ a \end{pmatrix} = 36 + 78a$	**M1 A1**	Takes dot product of the correct vectors
	$3\sqrt{30}\sqrt{36+26a^2} = \lvert 36 + 78a \rvert$	**M1**	Puts distance equal to $3\sqrt{30}$
	$\Rightarrow 15\left(18+13a^2\right) = \left(6+13a\right)^2$ $\Rightarrow 26\left(a-3\right)^2 = 0 \qquad \Rightarrow a = 3$	**A1**	
		5	

Q13

Question	Answer	Marks	Guidance
7(i)	Solve two equations $9 + 2\lambda = 7 + 2\mu$ $13 + 3\lambda = -2 - 3\mu$	**M1**	
	to obtain $\lambda = -3$ and $\mu = -2$.	**A1**	
	Use third equation to obtain $a = 2$.	**A1**	

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7(ii)	Normal to the plane is $\mathbf{n} = -12\mathbf{i} + 4\mathbf{k}$	**M1 A1**	
	Perpendicular distance $= \frac{1}{\sqrt{160}}\begin{pmatrix}-6\\6\\-8\end{pmatrix}\cdot\begin{pmatrix}-12\\0\\4\end{pmatrix} = \sqrt{10} = 3.16$ **Alt method:** Finds equation of plane M1, A1: Finds foot of perpendicular from P to plane M1 Hence length A1 **Alt method:** Find equation of plane $3x - z + 7 = 0$ M1, A1 Use formula $\frac{\lvert 3\times 3 + 0\times 1 - 6 + 7\rvert}{\sqrt{9+1}} = \sqrt{10}$ M1, A1	**M1 A1**	
7(iii)	Cross product of direction of $\boldsymbol{P}$ to l_2 with direction of l_2 $\begin{pmatrix}-6\\6\\-8\end{pmatrix}\times\begin{pmatrix}-1\\2\\-3\end{pmatrix} = \begin{pmatrix}-2\\-10\\-6\end{pmatrix}$	**M1 A1**	**Alt method:** Find N (foot of perpendicular) in terms of parameter and uses scalar product with n to find parameter M1, A1 so PN = $\begin{pmatrix}3\\0\\-1\end{pmatrix}$
	Perpendicular distance from $\boldsymbol{P}$ to l_2 is $\frac{\lvert 2\mathbf{i} + 10\mathbf{j} + 6\mathbf{k}\rvert}{\lvert -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\rvert} = \sqrt{10} = 3.16$	**M1 A1**	Find length PN M1, A1
		11	

Q14

Question	Answer	Marks	Guidance
8(i)	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\-4 & 1 & 3\\0 & 1 & 2\end{vmatrix} = -\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$	**M1**	Finds normal to $\boldsymbol{\Pi}_1$.
	$-x + 8y - 4z = 3$	**M1 A1**	Uses point on plane to find cartesian equation, AEF.
		3	

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8(ii)	$\begin{pmatrix}-1\\8\\-4\end{pmatrix}\cdot\begin{pmatrix}3\\1\\-1\end{pmatrix}=9\sqrt{11}\cos\theta$	**M1**	Uses scalar product.
	$\Rightarrow \theta = 72.5°$	**A1**	Accept 1.26 rad.
		2	
8(iii)	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\3 & 1 & -1\\-1 & 8 & -4\end{vmatrix}=4\mathbf{i}+13\mathbf{j}+25\mathbf{k}$	**M1 A1**	Finds direction of line of intersection.
	Point on both planes is, e.g. $(1,1,1)$.	**M1 A1**	Finds point common to both planes.
	$\mathbf{r}=\mathbf{i}+\mathbf{j}+\mathbf{k}+\lambda(4\mathbf{i}+13\mathbf{j}+25\mathbf{k})$	**A1**	States vector equation of line.
		5	

Q15

Question	Answer	Marks	Guidance
10(i)	$\overrightarrow{AB}=\begin{pmatrix}2\\3\\2\end{pmatrix}, \overrightarrow{CD}=\begin{pmatrix}1\\1.5\\1\end{pmatrix}=\frac{1}{2}\overrightarrow{AB}$	**B1**	Or shows if parallel, then m=3/2
		1	

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10(ii)	$\overrightarrow{AB}=\begin{pmatrix}2\\3\\2\end{pmatrix}$, $\overrightarrow{CD}=\begin{pmatrix}1\\m\\1\end{pmatrix}$ **and** $\overrightarrow{AC}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}$ **or** AD $=\begin{pmatrix}-1\\m-1\\1\end{pmatrix}$ **or** BC$=\begin{pmatrix}-4\\-4\\-2\end{pmatrix}$	**B1**	
	$\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&3&2\\1&m&1\end{vmatrix}=\begin{pmatrix}3-2m\\0\\2m-3\end{pmatrix}$ (so parallel to $\begin{pmatrix}1\\0\\-1\end{pmatrix}$)	**M1A1**	Finds common perpendicular using cross product.
	$\dfrac{\lvert AC.\boldsymbol{n}\rvert}{\lvert\boldsymbol{n}\rvert}=\dfrac{\lvert-2(3-2m)+0+0\rvert}{\sqrt{(3-2m)^2+(2m-3)^2}}$ o.	**M1**	Uses formula for shortest distance.
	$=\dfrac{2}{\sqrt{2}}=\sqrt{2}$	**A1**	
		5	
10(iii)	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&3&2\\-2&-1&0\end{vmatrix}=\begin{pmatrix}2\\-4\\4\end{pmatrix}\sim\begin{pmatrix}1\\-2\\2\end{pmatrix}$ o.	**M1A1**	Finds normal to plane ABC (AEF).
	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&3&2\\-1&1&1\end{vmatrix}=\begin{pmatrix}1\\-4\\5\end{pmatrix}$ o.	**A1**	Finds normal to ABD (AEF).
	$\cos\theta=\dfrac{1+8+10}{\sqrt{1^2+2^2+2^2}\sqrt{1^2+4^2+5^2}}\left(=\dfrac{19}{\sqrt{378}}\right)$	**M1A1FT**	Uses formula for angle between two lines.
	$\Rightarrow\theta=12.2°$	**A1**	CAO.
		6	

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Q16

Question	Answer	Marks	Guidance
12O(i)	$\overrightarrow{AC}=\begin{pmatrix}2\\-2\\-2\end{pmatrix}\quad \overrightarrow{AB}=\begin{pmatrix}2\\-2\\2\end{pmatrix}\quad \overrightarrow{CD}=\begin{pmatrix}2\\-4\\\alpha-1\end{pmatrix}$	**B1**	May find $\overrightarrow{AD}, \overrightarrow{BC}$ *or* $\overrightarrow{BD}$ instead of AC
	$\overrightarrow{AB}\times\overrightarrow{CD}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&1\\2&-4&\alpha-1\end{vmatrix}=\begin{pmatrix}5-\alpha\\3-\alpha\\-2\end{pmatrix}\sim\begin{pmatrix}\alpha-5\\\alpha-3\\2\end{pmatrix}$	**M1A1**	
	$\left\lvert\dfrac{\begin{pmatrix}2\\-2\\-2\end{pmatrix}\begin{pmatrix}\alpha-5\\\alpha-3\\2\end{pmatrix}}{\sqrt{(\alpha-5)^2+(\alpha-3)^2+4}}\right\rvert=2\sqrt{2}\Rightarrow 2\sqrt{2}\sqrt{2\alpha^2-16\alpha+38}=8$	**M1A1**	Substitutes their vectors into correct formula
		A1	
	$\Rightarrow(\alpha-3)(\alpha-5)=0\Rightarrow\alpha=3$ or 5 (AG)	**A1**	
	Total:	**7**	
12O(ii)	$\overrightarrow{AD}=\begin{pmatrix}4\\-6\\0\end{pmatrix}\quad\left(or\ \overrightarrow{CD}=\begin{pmatrix}2\\-4\\2\end{pmatrix}\right)$	**B1**	*Alt method:* Let P be point on AC with parameter $\boldsymbol{\lambda}$ **M1**
	Distance of D from $AC=\dfrac{1}{\sqrt{1+1+1}}\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&-6&0\\1&-1&-1\end{vmatrix}=\sqrt{\frac{56}{3}}=4.32$ (or with CD)	**M1A1**	Use DP.AC = 0 to find $\boldsymbol{\lambda}$ (= -5/3) **M1** Find length **A1**
	Total:	**3**	

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12O(iii)	ABC: $\mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & - \\ 1 & -1 & 1 \end{vmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ ABD: $\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & -3 & 0 \end{vmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$	**B1B1**	
	$\cos\theta = \left\lvert \frac{-3-2+0}{\sqrt{2}\sqrt{14}} \right\rvert \Rightarrow \theta = 19.1° \text{ or } 0.333 \text{ rads}$	**M1A1**	
	Total:	**4**	

Q17

Question	Answer	Marks	Guidance
9(i)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \begin{pmatrix} -7 \\ -1 \\ 3 \end{pmatrix}$	**M1A1**	
	Cartesian equation of Π_1 is $7x + y - 3z = \text{const}$	**M1**	
	$7\times1 + 2 - 3\times1 = 6 \Rightarrow 7x + y - 3z = 6$	**A1**	
	Total:	**4**	
9(ii)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -1 \\ 2 & 3 & -1 \end{vmatrix} = \begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix}$	**M1A1**	
	$5\times2 - (-3) + 7\times1 = 20 \Rightarrow 5x - y + 7z = 20$	**M1A1**	
	Total:	**4**	

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9(iii)	$\cos\theta = \left\lvert \dfrac{\begin{pmatrix}7\\1\\-3\end{pmatrix}.\begin{pmatrix}5\\-1\\7\end{pmatrix}}{\sqrt{49+1+9}\sqrt{25+1+49}} \right\rvert$	**M1M1**	
	$\Rightarrow \cos\theta = \dfrac{13}{\sqrt{59}\sqrt{75}} \Rightarrow \theta = 78.7°$ or 1.37 rad.	**A1**	
	Total:	**3**	

Q18

Question	Answer	Marks	Guidance
6(i)	$\overrightarrow{AB} = \mathbf{i}+5\mathbf{j}-2\mathbf{k}$ $\overrightarrow{BC} = -4\mathbf{i}-2\mathbf{j}+5\mathbf{k}$ $\overrightarrow{AC} = -3\mathbf{i}+3\mathbf{j}+3\mathbf{k}$	**B1**	2 correct required
	$\overrightarrow{AB}\times\overrightarrow{BC} = 21\mathbf{i}+3\mathbf{j}+18\mathbf{k}$ (*)	**M1A1**	OE
	Area of triangle $ABC = \frac{1}{2}\sqrt{21^2+3^2+18^2} = 13.9\left(\frac{3}{2}\sqrt{86}\right)$	**A1**	
	Alt method**:** Use scalar product to find angle	**(M1A1**	
	Find area using Area = ½ *ab* sin *C* or equivalent	**M1A1)**	
		4	

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6(ii)	$d = \dfrac{\left\lvert \overrightarrow{AB} \times \overrightarrow{BC} \right\rvert}{\left\lvert \overrightarrow{BC} \right\rvert} = \dfrac{\sqrt{21^2 + 3^2 + 18^2}}{\sqrt{4^2 + 2^2 + 5^2}}$	**M1A1**	Alt method: Find angle at C
	$= 4.15\left(\dfrac{1}{5}\sqrt{430}\right)$	**A1**	Area triangle = sin C × $\lvert AC \rvert$
	Alt method: Use equation of BC to find D (foot of perpendicular) in terms of parameter **and** scalar product to find parameter , λ= 8/15. Find length	**(M1A1)**	
		3	
6(iii)	From (*) Cartesian equation is $7x + y + 6z = \text{const.}$	**M1**	
	Through (2, –1 , 1) Hence $7x + y + 6z = 19$	**A1**	
		2	

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Q19

8

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -8 \\ -3 & 4 & -5 \end{vmatrix} = \begin{pmatrix} 17 \\ 34 \\ 17 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ **M1 A1**

So $x + 2y + z = \text{const} \Rightarrow \text{const} = 2 - 2 + 3 = 3 \left(\text{using a point}\right) \Rightarrow$ **M1**

$x + 2y + z = 3$ **A1**

[4]

$\sqrt{9+1+4}\sqrt{1+4+1}\cos\theta = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} . \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \Rightarrow \cos\theta = \frac{3}{\sqrt{14}\sqrt{6}} = \frac{3}{\sqrt{84}}$ **M1 M1**

$\Rightarrow \theta = 70.9°$ or 1.24 radians **A1**

[3]

Direction of line of intersection is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 3 & -1 & 2 \end{vmatrix} = \begin{pmatrix} 5 \\ 1 \\ -7 \end{pmatrix}$ **M1A1**

Finds point common to both planes is $\left(-1, 0, 4\right)$ or $\left(\frac{13}{7}, \frac{4}{7}, 0\right)$ or $(0, \frac{1}{5}, \frac{13}{5})$ **M1**

Equation of line of intersection is eg $\mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + t \begin{pmatrix} 5 \\ 1 \\ -7 \end{pmatrix}$. **A1**

[4]

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Qu	Soluti	Marks
11 (o) (i)	Direction perpendicular to AB and CD: $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 4 & 3-\lambda & 1 \end{vmatrix} = \begin{pmatrix} \lambda - 2 \\ 4 \\ -4 \end{pmatrix}$	M1A1
	$\overrightarrow{DB} = \begin{pmatrix} 5 \\ -4 \\ -6 \end{pmatrix}$. Hence $\left\lvert \frac{\begin{pmatrix} 5 \\ -4 \\ -6 \end{pmatrix} . \begin{pmatrix} \lambda - 2 \\ 4 \\ -4 \end{pmatrix}}{\sqrt{(\lambda-2)^2 + 16 + 16}} \right\rvert = 3$ or equivalent	M1A1
(ii)	$\Rightarrow (5\lambda - 2)^2 = 9(\lambda^2 - 4\lambda + 36)$ $\Rightarrow \ldots \Rightarrow \lambda^2 + \lambda - 20 = 0$ **(AG)**	M1M1 A1 [7]
	$(\lambda + 5)(\lambda - 4) = 0 \Rightarrow \lambda = -5, 4$	B1
	$\lambda = 4 \Rightarrow \mathbf{a} = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} \Rightarrow$ Normal to $ABD = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ -1 & 3 & 7 \end{vmatrix} = \begin{pmatrix} -10 \\ -29 \\ 11 \end{pmatrix}$	M1A1
	$\lambda = -5 \Rightarrow \mathbf{a} = \begin{pmatrix} 2 \\ -5 \\ -3 \end{pmatrix} \Rightarrow$ Normal to $ABD = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 8 & 1 \\ -1 & 12 & 7 \end{vmatrix} = \begin{pmatrix} 44 \\ -29 \\ 56 \end{pmatrix}$	A1
	$\cos\theta = \frac{-440 + 841 + 616}{\sqrt{10^2 + 29^2 + 11^2}\sqrt{44^2 + 29^2 + 56^2}} = \frac{1017}{\sqrt{1062}\sqrt{5913}}$	M1A1
	$\Rightarrow \theta = 66.1°$	A1 [7]

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Q21

11 E	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & -1 \\ 3 & -4 & -2 \end{vmatrix} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$	**M1A1**	
	Obtains three equations. E.g. from $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$ $t - 3s + 2k = 4$ $-3t + 4s - k = -2$ $-t + 2s + 5k = 4$	**M1A1**	
	Solves: $s = -1$, $t = -1$, $k = 1$	**M1A1**	
	$\mathbf{p} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{q} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ (Both required)	**A1**	[7]
	ALT METHOD: Find PQ in terms of s,t **M1** Calculate scalar products direction vectors for l_1 and l_2 **M1,A1,A1** Solve simultaneous equations **M1 A1** p and q **A1**		
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & - \\ 2 & -1 & 5 \end{vmatrix} = \begin{pmatrix} -22 \\ -19 \\ 5 \end{pmatrix}$	**B1**	
	$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -22 \\ -19 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$	**B1B1**	[3]

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	Direction of l_3 is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & - \\ 0 & 0 & 1 \end{vmatrix} = \begin{pmatrix} -4 \\ -3 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$	**B1**	
	Normal to Π is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 22 & 19 & -5 \\ 2 & -1 & 5 \end{vmatrix} = \begin{pmatrix} 90 \\ -120 \\ -60 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}$	**M1**	
	Cartesian equation of Π is $3x - 4y - 2z = 1$ (Since P lies in Π)	**A1**√	
	Vector equation of l_3 is $\mathbf{r} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$ (OE) (Since $z = 0$.)	**B1**	[4]

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Q22

110 (i)

$\mathbf{p} = \begin{pmatrix} 8+\lambda \\ 2-2\lambda \\ 3 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 5 \\ 3+2\mu \\ -14-3\mu \end{pmatrix} \Rightarrow \overrightarrow{QP} = \begin{pmatrix} 3+\lambda \\ -1-2\lambda-2\mu \\ 17+3\mu \end{pmatrix}$ M1A1

$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} . \begin{pmatrix} 3+\lambda \\ -1-2\lambda-2\mu \\ 17+3\mu \end{pmatrix} = 0 \Rightarrow 5\lambda + 4\mu = -5$ M1A1

$\begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} . \begin{pmatrix} 3+\lambda \\ -1-2\lambda-2\mu \\ 17+3\mu \end{pmatrix} = 0 \Rightarrow -4\lambda - 13\mu = 53$ A1

Solving : $\lambda = 3, \mu = -5$ M1A1

Whence: $\mathbf{p} = \begin{pmatrix} 11 \\ -4 \\ 3 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 5 \\ -7 \\ 1 \end{pmatrix}$. A1 (8)

(ii) (a)

$\overrightarrow{AP} = \begin{pmatrix} 3 \\ -6 \\ 0 \end{pmatrix}, \overrightarrow{AQ} = \begin{pmatrix} -3 \\ -9 \\ -2 \end{pmatrix}$ B1

$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & 0 \\ -3 & -9 & -2 \end{vmatrix} = \begin{pmatrix} 12 \\ 6 \\ -45 \end{pmatrix}$ (CAO) ; Area $= \frac{1}{2}\sqrt{2205}$ (=23.5) M1A1 A1

(b)

$\overrightarrow{AB} = \begin{pmatrix} -3 \\ 1 \\ -17 \end{pmatrix} \Rightarrow$ Volume $= \frac{1}{3} \times \frac{1}{2} \sqrt{2205} \left| \frac{\begin{pmatrix} -3 \\ 1 \\ -17 \end{pmatrix} . \begin{pmatrix} 12 \\ 6 \\ -45 \end{pmatrix}}{\sqrt{2205}} \right| = \frac{735}{6} (= 122.5)$ M1A1 (6)

Total: 14

Alternative: for marks 3,4,5,6 and 7 in part **(i)**

$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 0 \\ 0 & 2 & -3 \end{vmatrix} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$ (M1A1)

$6k - \lambda = 3$
$3k + 2\lambda + 2 = -1$
$2k - 3\mu = 17$ (A1)

Solving; $k = 1$, $\lambda = 3$ and $\mu = -5$ (If k missing, or assumed to be 1, deduct 1 mark.) (M1A1)

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8	$\begin{pmatrix}3+2\lambda\\ \lambda\\ 2-2\lambda\end{pmatrix}.\begin{pmatrix}1\\2\\1\end{pmatrix}=3$	M1
	$3+2\lambda+2\lambda+2-2\lambda=3\Rightarrow\lambda=-1$	A1
	P is $(1, -1, 4)$ (Accept position vector.)	A1 (3)
	$\mathbf{n}=\begin{pmatrix}1\\2\\1\end{pmatrix}$	B1
	$\mathbf{w}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&1\\2&1&-2\end{vmatrix}=\begin{pmatrix}-5\\4\\-3\end{pmatrix}$	M1A1 (3)
	Direction of PQ is $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-5&4&-3\\1&2&1\end{vmatrix}=\begin{pmatrix}10\\2\\-14\end{pmatrix}\sim\begin{pmatrix}5\\1\\-7\end{pmatrix}$ (OE)	M1A1
	Equation of PQ is $\mathbf{r}=\begin{pmatrix}1\\-1\\4\end{pmatrix}+\mu\begin{pmatrix}5\\1\\-7\end{pmatrix}$	dM1A1 (4)
	[N.B. For methods not using **w** at most three B1 marks.]	**Total 10**

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11 E	E. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ -1 & 0 & 4 \end{vmatrix} = \begin{pmatrix} 8 \\ 4 \\ 2 \end{pmatrix} \sim \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$	**M1A1**
	$\dfrac{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}}{\sqrt{4^2+2^2+1^2}} = \dfrac{4}{\sqrt{21}}$ (AG)	**M1A1** [4]
	$\mathbf{p} = \dfrac{3}{\sqrt{21}}\left(\dfrac{4\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{21}}\right) = \dfrac{1}{7}(4\mathbf{i}+2\mathbf{j}+\mathbf{k})$	**B1**
	Line AP: $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$	**M1A1**
	For Q $1-3t=0 \Rightarrow t = \dfrac{1}{3} \Rightarrow \mathbf{q} = \dfrac{1}{3}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$	**M1A1** [5]
	E. $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, $\overrightarrow{BQ} = \dfrac{1}{3}\begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix}$	**B1**
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ 0 & -4 & 1 \end{vmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$	**M1A1**
	$\cos^{-1}\left\lvert\dfrac{\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}.\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}}{\sqrt{21}.\sqrt{21}}\right\rvert = \cos^{-1}\dfrac{8+2+4}{21} = \cos^{-1}\dfrac{14}{21} = \cos^{-1}\dfrac{2}{3}$ (AG)	**M1A1** [5] **Total 14**

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Q25

11	$\overrightarrow{AB} = \begin{pmatrix} 6 \\ 4 \\ -8 \end{pmatrix}$ $\overrightarrow{CD} = \begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix}$	B1
	Common perpendicular is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -4 \\ 5 & -2 & -4 \end{vmatrix} = \begin{pmatrix} -16 \\ -8 \\ -16 \end{pmatrix} \sim \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$	M1A1
	$\overrightarrow{AD} = \begin{pmatrix} 1 \\ -4 \\ 9 \end{pmatrix} \Rightarrow$ shortest distance $= \dfrac{\left\lvert \begin{pmatrix} 1 \\ -4 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\rvert}{\sqrt{4+1+4}} = \dfrac{16}{3}$ or 5.33	M1A1 [5]
	Normal to Π_1 is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 9 \\ 3 & 2 & -4 \end{vmatrix} = \begin{pmatrix} -2 \\ 31 \\ 14 \end{pmatrix}$	M1A1
	Normal to Π_2 is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 9 \\ 5 & -2 & -4 \end{vmatrix} = \begin{pmatrix} 34 \\ 49 \\ 18 \end{pmatrix}$	A1
	$\cos\theta = \dfrac{-2\times 34 + 31\times 49 + 14\times 18}{\sqrt{2^2+31^2+14^2}\sqrt{34^2+49^2+18^2}}$	M1A1√
	$\Rightarrow \theta = 36.7°$ (CAO)	A1 [6]

Q26

	OR	
(i)	$\mathbf{r} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} + 7\mathbf{j} + \mathbf{k}) + \mu(3\mathbf{i} + \mathbf{j} - \mathbf{k}) \Rightarrow A$ is in Π_1.	B1 [1]
(ii)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 7 & 1 \\ 3 & 1 & -1 \end{vmatrix} = \begin{pmatrix} -8 \\ 4 \\ -20 \end{pmatrix} \sim \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$	M1A1 [2]
(iii)	L is (12, –6, 6)	B1
	$2x - y + 5z = 24 + 6 + 30 = 60$	B1
	$\mathbf{n} = t(2\mathbf{i} - \mathbf{j} + 5\mathbf{k}) \Rightarrow 4t + t + 25t = 60 \Rightarrow t = 2$	M1A1√
	$\mathbf{n} = 4\mathbf{i} - 2\mathbf{j} + 10\mathbf{k}$	A1 [5]
(iv)	$\mathbf{m} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + \dfrac{3}{4}(8\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}) = 10\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$ (AG)	B1
	M is (10, –5, 5) $\Rightarrow \overrightarrow{NM} = 6\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$	B1√
	$(6\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} + 5\mathbf{k}) = -\mathbf{i} + 2\mathbf{j}$	M1A1
	Perpendicular distance is $\dfrac{\lvert 20(-\mathbf{i} + 2\mathbf{j} \rvert}{\sqrt{30}} = \dfrac{20}{\sqrt{6}} = 8.16$	M1A1
	(Mark various alternative methods in a similar manner.)	[6]

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Q27

10 (i)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -3 \\ -2 & 1 & 3 \end{vmatrix} = \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix} \sim \begin{pmatrix} \mathbf{1} \\ \mathbf{-1} \\ \mathbf{1} \end{pmatrix}$	M1A1
	$\overrightarrow{BA} = 6\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$	B1
	Shortest distance $\left\lvert \frac{\begin{pmatrix} 6 \\ 4 \\ -6 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}{\sqrt{1^2+1^2+}} \right\rvert = \frac{4}{\sqrt{3}}$ $(= 2.31)$	M1A1 (5)
	Alternative	
	$\begin{pmatrix} -6-2\lambda-\mu \\ -4+\lambda+2\mu \\ 6+3\lambda+3\mu \end{pmatrix} . \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = 0 \Rightarrow -13\lambda - 14\mu = 16$	(M1)
	$\begin{pmatrix} -6-2\lambda-\mu \\ -4+\lambda+2\mu \\ 6+3\lambda+3\mu \end{pmatrix} . \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = 0 \Rightarrow 14\lambda + 113\mu = -26$	(A1)
	$\Rightarrow \lambda = -\frac{52}{9},\ \mu = \frac{38}{9}$	(M1A1)
	Shortest distance $= \frac{4}{\sqrt{3}}$. (= 2.31	(A1)
(ii)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ -2 & 1 & 3 \end{vmatrix} = \begin{pmatrix} -4 \\ -5 \\ -1 \end{pmatrix} \sim \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$	M1A1
	Cartesian equation of Π: $4x + 5y + z = -12 - 5 + 2 = -15$	M1A1 (4)
(iii)	Distance of A from Π: $\left\lvert \frac{\begin{pmatrix} 6 \\ 4 \\ -6 \end{pmatrix} . \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}}{\sqrt{4^2+5^2+1^2}} \right\rvert$	M1A1
	$= \frac{38}{\sqrt{42}}$ $(= 5.86)$	A1 (3) **[12]**

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11 O	Vector perpendicular to Π_1	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 1 & -3 & 1 \end{vmatrix} = \begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix}$	M1A1		
	Obtains cartesian equation	$3\times 2+1\times(-1)+(-2)\times(-6)=17$ $\Rightarrow 3x-y-6z=17$	M1 A1	4	
	Obtains area of triangle *ABC*.	$\frac{1}{2}\sqrt{3^2+1^2+6^2}=\frac{1}{2}\sqrt{46}$ (=3.39)	M1 A1		
	Obtains length of perpendicular from *D* to triangle *ABC*.	$\left\lvert\frac{9-6-12-17}{\sqrt{3^2+1^2+6^2}}\right\rvert=\frac{26}{\sqrt{46}}$	M1A1		
	Uses $\frac{1}{3}\times$ Base area $\times$ Height.	Either $\frac{1}{3}\times\frac{1}{2}\sqrt{46}\times\frac{26}{\sqrt{46}}=\frac{13}{3}$	M1A1		
	or triple scalar product method.	Or e.g. $\left\lvert\frac{1}{6}\begin{pmatrix}1\\5\\4\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\-6\end{pmatrix}\right\rvert=\frac{26}{6}=\frac{13}{3}$		6	
	Obtains normal to *ABD*.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 5 & 4 \\ 2 & 0 & 1 \end{vmatrix} = \begin{pmatrix} 5 \\ 7 \\ -10 \end{pmatrix}$	M1A1		
	Uses scalar product	$\sqrt{3^2+1^2+6^2}\sqrt{5^2+7^2+10^2}\cos\theta=\begin{pmatrix}3\\-1\\-6\end{pmatrix}\cdot\begin{pmatrix}5\\7\\-10\end{pmatrix}$	M1		
	to find angle between normals and hence angle between Π_1 and Π_2.	$\Rightarrow\cos\theta=\frac{68}{\sqrt{46}\sqrt{174}}\Rightarrow\theta=40.5^{\circ}$	A1		**[14]**

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Q29

11 E	Finds **PQ**.	$\mathbf{PQ} = \begin{pmatrix} -3+\mu-3\lambda \\ -6\mu-2\lambda \\ 12-2\mu+\lambda \end{pmatrix}$	M1A1		
	Finds direction of common perpendicular.	$\begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ 3 & 2 & -1 \\ 1 & -6 & -2 \end{vmatrix} = \begin{pmatrix} -10 \\ 5 \\ -20 \end{pmatrix} \sim \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$	M1		
	(Or uses scalar product of **PQ** with the direction vector of each line.)				
	Obtains two correct equations. e.g.	$-3+\mu-3\lambda = 12\mu+4\lambda$ $-24\mu-8\lambda = -12+2\mu-\lambda$	A1 A1		
	Solves.	$\mu = 1,\ \lambda = -2$	M1A1	8	
	Finds **p** and **q**.	$\mathrm{p} = \begin{pmatrix} 4+3\lambda \\ 7+2\lambda \\ -1-\lambda \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$ $\mathrm{q} = \begin{pmatrix} 1+\mu \\ 7-6\mu \\ 11-2\mu \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix}$	A1		
	Finds common perpendicular.	$\mathbf{AB} \times \mathbf{PQ} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ -1 & 0 & 4 \\ 2 & -1 & 4 \end{vmatrix} = \begin{pmatrix} 4 \\ 12 \\ 1 \end{pmatrix}$	M1A1		
	Finds **PA** (or **QA** or **PB** or **QB**)	$\mathbf{PA}\begin{pmatrix} 6 \\ 4 \\ -2 \end{pmatrix}$ $\mathbf{QA}\begin{pmatrix} 2 \\ 6 \\ -10 \end{pmatrix}$ $\mathbf{PB}\begin{pmatrix} 3 \\ 4 \\ 10 \end{pmatrix}$ $\mathbf{QB}\begin{pmatrix} -1 \\ 6 \\ 2 \end{pmatrix}$	B1		
	Uses triple scalar product to find shortest distance.	$\dfrac{\begin{pmatrix} 6 \\ 4 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 12 \\ 1 \end{pmatrix}}{\sqrt{16+144+1}} = \dfrac{24+48-2}{\sqrt{161}} = \dfrac{70}{\sqrt{161}} = 5.52$	M1A1 A1	6	**[14]**
	Alternative for last 4 marks:	Plane through e.g. P in direction **PA :** $4x + 12y - 29 = 0$. Award M1A1. Then use of distance of point from line formula to get $\dfrac{70}{\sqrt{161}}$. Award M1A1.			

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Q30

8	Finds normal to Π_1.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & -1 & -2 \end{vmatrix} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$	M1A1		
	Finds Cartesian equation.	Equation of Π_1 : $x + 3y - z = 12$	A1	3	
	Finds angle between normals, using scalar product.	$\cos\theta = \left\|\frac{2-3-1}{\sqrt{11}\sqrt{6}}\right\|$ $= \frac{2}{\sqrt{66}} \Rightarrow \theta = 75.7°$ or 1.32 rad.	M1 A1	2	
	Finds direction of line of intersection, using vector product.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & -1 & 1 \end{vmatrix} = 2\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}$	M1A1		
	Finds point common to both planes.	Point on both planes is e.g. (6,2,0)	M1A1		
	States vector equation.	$\mathbf{r} = 6\mathbf{i} + 2\mathbf{j} + t(2\mathbf{i} - 3\mathbf{j} - 7\mathbf{k})$ (OE)	A1	5	**[10]**

Q31

8	Finds normal	$\overrightarrow{AB} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{AC} = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$			
	and cartesian equation	$\overrightarrow{AB} \times \overrightarrow{AC} = -6\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$	M1A1		
		$6x + 2y - 5z = \text{constant} = 24 + 10 - 30 = 4$	M1A1	**(4)**	
	Finds where *OD* meets plane	Equation of *OD*: $\mathbf{r} = t(6\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})$	B1		
		$\Rightarrow 36t + 6t - 30t = 12t = 4 \Rightarrow t = \frac{1}{3}$	M1A1		
		E is the point (2,1,2).	A1	**(4)**	
	Obtains angles between planes	Using $(6\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}).\, 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$			
		$\Rightarrow 12 + 2 - 10 = \sqrt{36 + 4 + 25}\sqrt{4 + 1 + 4}\sin\theta$	M1A1		
		$\Rightarrow \theta = 9.5°$ (0.166 rad).	A1	**(3)**	**[11]**

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Q32

11	**OR** Obtains direction of common perpendicular.	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ 0 & 1 & m-1 \end{vmatrix} = -m\mathbf{i} + 4(1-m)\mathbf{j} + 4\mathbf{k}$	M1A1		
	Uses result for shortest distance between lines.	$\dfrac{\left\lvert \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} \begin{pmatrix} -m \\ 4-4m \\ 4 \end{pmatrix} \right\rvert}{\sqrt{m^2 + 16(1-2m+m^2)+16}} = 3$	M1A1		
	Solves equation.	$\Rightarrow \ldots \Rightarrow 19m^2 - 40m + 4 = 0$	A1		
		$\Rightarrow (19m-2)(m-2) = 0$	M1		
		$\Rightarrow m = 2$, since m is an integer. (AG)	A1	7	
	Finds relevant vectors.	$\mathbf{CA} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$ and $\mathbf{CD} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ or $\mathbf{AD} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$	B1		
	Use of cross-product.	$\dfrac{1}{\sqrt{17}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 1 & 0 & -4 \end{vmatrix} = \dfrac{1}{\sqrt{17}} \begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}$	M1		
	Obtains shortest distance.	$\dfrac{1}{\sqrt{17}} \sqrt{4^2 + 1^2 + 1^2} = \sqrt{\dfrac{18}{17}}$ (= 1.03)	A1	3	
	Finds 2nd vector in BCD (**CD** may already have been found.)	$\mathbf{BC} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$	B1		
	Finds normal vector to BCD. (Normal to ACD already found.)	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 5 \\ 0 & 1 & 1 \end{vmatrix} = -6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} \sim 2\mathbf{i} + \mathbf{j} - \mathbf{k}$	M1		
	Finds angle between planes = angle between normal vectors.	$\cos\theta = \dfrac{(4\mathrm{i} - \mathrm{j} + \mathrm{k}) \cdot (2\mathrm{i} + \mathrm{j} - \mathrm{k})}{\sqrt{16+1+1}\sqrt{4+1+1}} = \dfrac{6}{\sqrt{18}\sqrt{6}} = \dfrac{1}{\sqrt{3}}$	M1		
		$\therefore$ Angle between planes $= \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ (AG)	A1	4	**[14]**

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OR

11	**OR** **Alternatives for middle part:**				
	Or (a) Vector from D to any point on AC	$\begin{pmatrix} 1+t \\ -1 \\ -5-4t \end{pmatrix}$	(B1)		
	Uses orthogonality to obtain t.	$\begin{pmatrix} 1+t \\ -1 \\ -5-4t \end{pmatrix}.\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}=0 \Rightarrow t=-\frac{21}{17}$	(M1)		
	Finds magnitude of perpendicular.	$\frac{1}{\sqrt{17}}\sqrt{4^2+1^2+1^2}=\sqrt{\frac{18}{17}}$ (= 1.03)	(A1)	(3)	
	Or (b) Finds length of AD (or CD)	$\left\lvert\overrightarrow{AD}\right\rvert=\sqrt{27}$	(B1)		
	Finds projection of AD (or CD) onto AC.	$\left\lvert\frac{\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}.\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}}{\sqrt{4^2+1^2}}\right\rvert=\frac{21}{\sqrt{17}}$	(M1)		
	Finds perpendicular by Pythagoras.	$\sqrt{27-\frac{441}{17}}=\sqrt{\frac{18}{17}}$ (= 1.03)	(A1)	(3)	

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9	Finds vector normal to Π.	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 2 \\ 3 & 1 & -2 \end{vmatrix} = 2\mathbf{i} + 8\mathbf{j} + 7\mathbf{k}$	M1A1		
	Dot product of this with general point on l_1.	$\begin{pmatrix} 3+8t \\ 6+5t \\ 12-8t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 8 \\ 7 \end{pmatrix} = 138$ or $\begin{pmatrix} 8 \\ 5 \\ -8 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 8 \\ 7 \end{pmatrix} = 0$	A1√		
	Deduces result.	Independent of $t \Rightarrow$ parallel, or $\Rightarrow$ parallel.	A1	4	
	Cartesian equation of Π.	Π: $2x + 8y + 7z = 21$	B1		
	Substitutes general point of l_2.	Sub. $x = 5 + 2s$, $y = -4 - s$, $z = 7 + s$	M1		
	Finds value of parameter.	$\Rightarrow s = -2$	A1		
	Finds p.v. of intersection.	and line meets Π at point with p.v. $\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$	A1	4	
		Take (9,11,2) as A, (3,6,12) as B and let C be foot of perpendicular from A to l.			
		$\text{AB} = \sqrt{6^2 + 5^2 + 10^2} = \sqrt{161}$	B1		
		$\text{BC} = \dfrac{1}{\sqrt{(6^2\,5^2 + 8^2)(6\text{i} + 5\text{j} - 10\text{k})(8\text{i} + 5\text{j} - 8\text{k})}} = \dfrac{153}{\sqrt{153}} = \sqrt{153}$	M1		
	Finds distance from point to known point on l.		A1	4	**[12]**
	Finds distance along l from known point to foot of perpendicular from given point to l.	$\text{AC} = \sqrt{161 - 153} = \sqrt{8}$ or $2\sqrt{2}$ (= 2.83)	A1√		
		Alternatively:			
		$5 + 2s = 2 + \lambda + 3\mu$ $-4 - s = 3 - 2\lambda + \mu$ $7 + s = -1 + 2\lambda - 2\mu$	(B1)		
	F.t. on non-hypotenuse side (must be real).	$\Rightarrow s = -2$, $\Rightarrow \lambda = 2$, $\mu = -1$	(M1A1)		
	Writes a set of three equations in three unknowns for the intersection of l with Π.	and line meets Π at point with p.v. $\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$	(A1)		
	Solves the set of equations.				
	Finds p.v. of intersection.				

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Q33

9	Finds vector $\overrightarrow{BA}$.	$\overrightarrow{BA} = 6\mathbf{i} + 5\mathbf{j} - 10\mathbf{j}$	B1		
		$\left\|\frac{1}{\sqrt{8^2+5^2+8}}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 6 & 5 & -10\\ 8 & 5 & -8\end{vmatrix}\right\|$	M1A1		
		$=\sqrt{\frac{1224}{153}} = \sqrt{8}$ or $2\sqrt{2}$ $(=2.83)$	A1	(4)	
		Alternatively: (A)			
	Finds distance from point to known point on *l*.	Take (9,11,2) as *A*, (3,6,12) as *B* and let *C* be foot of perpendicular from *A* to *l*.			
		$AB = \sqrt{6^2+5^2+10^2} = \sqrt{161}$	(B1)		
	Finds distance along *l* from known point to foot of perpendicular from given point to *l*.	$BC = \frac{1}{\sqrt{8^2+5^2+8^2}}(6\mathbf{i}+5\mathbf{j}-10\mathbf{k})(8\mathbf{i}+5\mathbf{j}-8\mathbf{k})$	(M1)		
	F.t. on non-hypotenuse si (must be real).	$=\frac{153}{\sqrt{153}} = \sqrt{153}$	(A1) (A1ft)	(4)	
		$AC = \sqrt{161-153} = \sqrt{8}$ or $2\sqrt{2}$ $(=2.83)$			
		(B)			
	Finds vector . $\overrightarrow{AC}$	$\overrightarrow{AC} = \begin{pmatrix}8t-6\\5t-5\\10-8t\end{pmatrix}$	(B1)		
	Uses $\overrightarrow{AC}$ perpendicular to *l* to find *t*.	$\begin{pmatrix}8t-6\\5t-5\\10-8t\end{pmatrix}.\begin{pmatrix}8\\5\\-8\end{pmatrix} = 0 \Rightarrow t = 1$	(M1A1)		
	Finds length *AC*.	$AC = \sqrt{2^2+0^2+2^2} = \sqrt{8}$	(A1)	(4)	**[12]**

Q34

4	Finds vector product.	$\overrightarrow{AB} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix}1\\1\\2\end{pmatrix}$	B1		
		$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 3\\ 1 & 1 & 2\end{vmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$	M1A1	3	
	Finds area of triangle.	Area of triangle $ABC = \frac{1}{2}\left\|\begin{pmatrix}1\\1\\-1\end{pmatrix}\right\| = \frac{1}{2}\sqrt{3}$	M1A1		
	Finds length of perpendicular.	$\frac{1}{2}\sqrt{1^2+2^2+3^2}\,d = \frac{1}{2}\sqrt{3} \Rightarrow d = \sqrt{\frac{3}{14}}$	A1	3	**[6]**

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Q35

9	Finds normal vector to plane.	$\begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ 1 & -2 & -1 \\ 1 & 2 & -2 \end{vmatrix} = \begin{pmatrix} 6 \\ 1 \\ 4 \end{pmatrix}$ or $x = 2 + \lambda + \mu$, $y = -3 - 2\lambda + 2\mu$, $z = 1 - \lambda - 2\mu$	M1A1		
	Uses known point to find constant term.	Plane equation is $6x + y + 4z =$ constant. Substitute $(2, -3, 1) \Rightarrow 12 - 3 + 4 = 13$. Or eliminate λ and μ. $\Rightarrow \Pi_1 : 6x + y + 4z = 13$	M1 A1	4	
	Angle between normals is equal to angle between planes.	$\cos\theta = \dfrac{(6\mathrm{i} + \mathrm{j} + 4\mathrm{k}).(3\mathrm{i} - 2\mathrm{j} - 3\mathrm{k})}{\sqrt{6^2 + 1^2 + 4^2}\sqrt{3^2 + 2^2 + 3^2}} = \dfrac{4}{\sqrt{53}\sqrt{22}}$ $\Rightarrow \theta = 83.3^\circ$ or 1.45 rad.	M1A1 A1	3	
	Solve plane equation simultaneously.	$6x + y + 4z = 13$ and $3x - 2y - 3z = 4$ Obtains e.g. $y + 2z = 1$ and $3x + z = 6$ Or two of $(0, -11, 6)$, $(11/6, 0, 1/2)$, $(2, 1, 0)$	M1 A1A1		
	(Note: may find direction from vector product and use with one point.)	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ -6 \\ 3 \end{pmatrix}$ (OE)	A1	4	
		Alternatively: Direction of line from vector product. Finds a point on line. States equation of line.	 (M1A1) (A1) (A1)		**[11]**

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PROOF BY INDUCTION

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Q1

1 Prove by mathematical induction that $2^{4n} + 31^n - 2$ is divisible by 15 for all positive integers n. [6]

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Q2

3 **(a)** Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n}\left(5r^4+r^2\right)=\tfrac{1}{2}n^2(n+1)^2(2n+1).$$ [6]

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(b) Use the result given in part **(a)** together with the List of formulae (MF19) to find $\sum_{r=1}^{n} r^4$ in terms of n, fully factorising your answer. [3]

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Q3

3 The sequence of real numbers $a_1, a_2, a_3, \ldots$ is such that $a_1 = 1$ and

$$a_{n+1} = \left(a_n + \frac{1}{a_n}\right)^3.$$

(a) Prove by mathematical induction that $\ln a_n \geqslant 3^{n-1} \ln 2$ for all integers $n \geqslant 2$. [6]

[You may use the fact that $\ln\left(x + \frac{1}{x}\right) > \ln x$ for $x > 0$.]

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(b) Show that $\ln a_{n+1} - \ln a_n > 3^{n-1} \ln 4$ for $n \geqslant 2$. [2]

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Q4

2 It is given that $y = xe^{ax}$, where a is a constant.

Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n y}{dx^n} = \left(a^n x + na^{n-1}\right)e^{ax}. \qquad [6]$$

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Q5

5 Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}}(x\sin x) = (-1)^{n-1}\left(x\cos x + (2n-1)\sin x\right). \qquad [7]$$

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Q6

2 Prove by mathematical induction that $7^{2n} - 1$ is divisible by 12 for every positive integer n. [5]

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Q7

2 The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for $n \geqslant 1$.

(a) Prove by induction that $u_n = 2^n - 1$ for all positive integers n. [5]

(b) Deduce that u_{2n} is divisible by u_n for $n \geqslant 1$. [2]

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Q8

2 It is given that $y = \ln(ax + 1)$, where a is a positive constant. Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = (-1)^{n-1} \frac{(n-1)!a^n}{(ax+1)^n}. \qquad [6]$$

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Q9

8 **(i)** Prove by mathematical induction that, for $z \neq 1$ and all positive integers n,

$$1 + z + z^2 + \ldots + z^{n-1} = \frac{z^n - 1}{z - 1}. \qquad [5]$$

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Q10

1 Prove by mathematical induction that $3^{3n} - 1$ is divisible by 13 for every positive integer n. [5]

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Q11

2 It is given that $f(n) = 2^{3n} + 8^{n-1}$. By simplifying $f(k) + f(k + 1)$, or otherwise, prove by mathematical induction that $f(n)$ is divisible by 9 for every positive integer n. [6]

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Q12

9 For the sequence $u_1, u_2, u_3, \ldots$, it is given that $u_1 = 8$ and

$$u_{r+1} = \frac{5u_r - 3}{4}$$

for all r.

(i) Prove by mathematical induction that

$$u_n = 4\left(\tfrac{5}{4}\right)^n + 3,$$

for all positive integers n. [5]

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(ii) Deduce the set of values of x for which the infinite series

$$(u_1 - 3)x + (u_2 - 3)x^2 + \ldots + (u_r - 3)x^r + \ldots$$

is convergent. [2]

(iii) Use the result given in part **(i)** to find surds a and b such that

$$\sum_{n=1}^{N} \ln(u_n - 3) = N^2 \ln a + N \ln b.$$ [3]

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Q13

3 The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 < 3$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{4u_n + 9}{u_n + 4}.$$

(i) By considering $3 - u_{n+1}$, or otherwise, prove by mathematical induction that $u_n < 3$ for all positive integers n. [5]

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(ii) Show that $u_{n+1} > u_n$ for $n \geqslant 1$. [3]

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Q14

6 It is given that $y = e^x u$, where u is a function of x. The rth derivatives $\frac{d^r y}{dx^r}$ and $\frac{d^r u}{dx^r}$ are denoted by $y^{(r)}$ and $u^{(r)}$ respectively. Prove by mathematical induction that, for all positive integers n,

$$y^{(n)} = e^x\left(\binom{n}{0}u + \binom{n}{1}u^{(1)} + \binom{n}{2}u^{(2)} + \ldots + \binom{n}{r}u^{(r)} + \ldots + \binom{n}{n}u^{(n)}\right). \quad [8]$$

[You may use without proof the result $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$.]

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Q15

2 Prove, by mathematical induction, that $5^n + 3$ is divisible by 4 for all non-negative integers n. [5]

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Q16

3 Prove, by mathematical induction, that $\sum_{r=1}^{n} r \ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(n+1)^n}{n!}\right)$ for all positive integers n. [6]

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Q17

3 **(i)** Show that $\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}(x^{n+1}\ln x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^n + (n+1)x^n \ln x\right)$. [2]

(ii) Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n \ln x) = n!\left(\ln x + 1 + \frac{1}{2} + \ldots + \frac{1}{n}\right).$$ [5]

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Q18

3 Prove by mathematical induction that, for all positive integers n, $10^n + 3 \times 4^{n+2} + 5$ is divisible by 9. [6]

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Q19

2 It is given that a diagonal of a polygon is a line joining two non-adjacent vertices. Prove, by mathematical induction, that an n-sided polygon has $\frac{1}{2}n(n-3)$ diagonals, where $n \geqslant 3$. [6]

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Q20

4 Using factorials, show that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$. [2]

Hence prove by mathematical induction that

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \ldots + \binom{n}{r}a^{n-r}x^r + \ldots + \binom{n}{n}x^n$$

for every positive integer n. [4]

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3 The sequence $a_1, a_2, a_3, \ldots$ is such that $a_1 > 5$ and $a_{n+1} = \frac{4a_n}{5} + \frac{5}{a_n}$ for every positive integer n.

Prove by mathematical induction that $a_n > 5$ for every positive integer n. [5]

Prove also that $a_n > a_{n+1}$ for every positive integer n. [2]

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Q22

3 Prove by mathematical induction that, for all positive integers n, $\sum_{r=1}^{n} \frac{1}{(2r)^2 - 1} = \frac{n}{2n+1}$. [6]

State the value of $\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1}$. [1]

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3 Given that a is a constant, prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x\mathrm{e}^{ax}) = na^{n-1}\mathrm{e}^{ax} + a^n x\mathrm{e}^{ax}. \qquad [6]$$

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3 Prove by mathematical induction that, for all non-negative integers n,

$$11^{2n} + 25^n + 22$$

is divisible by 24. [6]

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3 It is given that $\phi(n) = 5^n(4n + 1) - 1$, for $n = 1, 2, 3, \ldots$. Prove, by mathematical induction, that $\phi(n)$ is divisible by 8, for every positive integer n. [7]

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3 It is given that $u_r = r \times r!$ for $r = 1, 2, 3, \ldots$. Let $S_n = u_1 + u_2 + u_3 + \ldots + u_n$. Write down the values of

$$2! - S_1, \quad 3! - S_2, \quad 4! - S_3, \quad 5! - S_4. \qquad [2]$$

Conjecture a formula for S_n. [1]

Prove, by mathematical induction, a formula for S_n, for all positive integers n. [4]

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Q27

2 Prove by mathematical induction that $5^{2n} - 1$ is divisible by 8 for every positive integer n. [5]

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3 Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^x \sin x) = (\sqrt{2})^n \mathrm{e}^x \sin(x + \tfrac{1}{4}n\pi).$$ [7]

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Q29

5 It is given that $y = (1 + x)^2 \ln(1 + x)$. Find $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3}$. [3]

Prove by mathematical induction that, for every integer $n \geqslant 3$,

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = (-1)^{n-1} \frac{2(n-3)!}{(1+x)^{n-2}}.$$ [5]

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[Turn over

9 Prove by mathematical induction that, for every positive integer n,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta. \quad [5]$$

Express $\sin^5\theta$ in the form $p\sin 5\theta + q\sin 3\theta + r\sin\theta$, where p, q and r are rational numbers to be determined. [6]

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2 Prove, by mathematical induction, that, for integers $n \geqslant 2$,

$$4^n > 2^n + 3^n.$$ [5]

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Q32

5 Let I_n denote $\int_0 x^n e^{-2x}\,dx$. Show that $I_n = \frac{1}{2}nI_{n-1}$, for $n \geqslant 1$. [2]

Prove by mathematical induction that, for all positive integers n, $I_n = \dfrac{n!}{2^{n+1}}$. [6]

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[Turn over

Q33

3 Let $S_N = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{N}{(N+1)!}$. Prove by mathematical induction that, for all positive integers N,

$$S_N = 1 - \frac{1}{(N+1)!}.$$ [5]

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Q34

2 For the sequence $u_1, u_2, u_3, \ldots$, it is given that $u_1 = 1$ and $u_{r+1} = \dfrac{3u_r - 2}{4}$ for all r. Prove by mathematical induction that $u_n = 4\left(\frac{3}{4}\right)^n - 2$, for all positive integers n. [5]

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PROOF BY INDUCTION
MS

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Q1

Question	Answer	Marks	Guidance
1	$2^4+31-2=45$ is divisible by 15	**B1**	Checks base case.
	Assume that $2^{4k}+31^k-2$ is divisible by 15 for some positive integer k.	**B1**	States inductive hypothesis.
	Then $2^{4k+4}+31^{k+1}-2=(15+1)2^{4k}+(30+1)31^k-2$	**M1 A1**	Separates $2^{4k}+31^k-2$ or considers difference.
	is divisible by 15 because $15\times2^{4k}+30\times31^k$ is divisible by 15.	**A1**	
	Hence, by induction, true for every positive integer n.	**A1**	
		6	

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Q2

Question	Answer	Marks	Guidance
3(a)	$5\times1^4+1^2=\frac{1}{2}(2)^2(2+1)[=6]$ so H_1 is true.	**B1**	Checks base case.
	Assume that $\sum_{r=1}^{k}\left[\left(5r^4+r^2\right)\right]=\frac{1}{2}k^2(k+1)^2(2k+1)$	**B1**	States inductive hypothesis [for some k] including the algebraic form. If says for **ALL** k, then B0.
	$\sum_{r=1}^{k+1}\left[\left(5r^4+\quad\right)\right]=\frac{1}{2}k\ (k+1)\ (2k+1)+5(k+1)\ +(k+1)$	**M1**	Considers sum to $k+1$.
	$\frac{1}{2}(k+1)^2\left(2k^3+k^2+10(k+1)^2+2\right)$	**M1**	Take out the factor of $(k+1)^2$ OR expands the summation expression and the target expression for $k+1$ and collects like terms for both.
	$\frac{1}{2}(k+1)^2(2k^3+11k^2+20k+12)=\frac{1}{2}(k+1)^2(k+2)^2(2k+3)$	**A1**	Factorises or having expanded, checks explicitly. At least one intermediate step seen following the award of M1 before reaching the answer.
	So H_{k+1} is true. By induction, H_n is true for all positive integers n.	**A1**	States conclusion. Implication must be clearly expressed.
		6	
3(b)	$5\sum_{r=1}^{n}r^4+\frac{1}{6}n(n+1)(2n+1)=\frac{1}{2}n^2(n+1)^2(2n+1)$	**M1**	Uses correct formula for $\sum r^2$.
	$[5]\sum_{r=1}^{n}r^4=\frac{1}{6}n(n+1)(2n+1)(3n(n+1)-1)$	**M1**	Makes $\sum r^4$ the subject and takes out all linear factors and the remaining term is of correct form.
	$\sum_{r=1}^{n}r^4=\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$	**A1**	CAO
		3	

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Q3

Question	Answer	Marks	Guidance
3(a)	$a_2 = 2^3$ leading to $\ln a_2 = 3\ln 2$ so true when $n = 2$.	**B1**	Checks base case. Must be n =2 and exact.
	Assume that $\ln a_k \geqslant 3^{k-1} \ln 2$.	**B1**	States inductive hypothesis.
	$\ln a_{k+1} = 3\ln\left(a_k + \frac{1}{a_k}\right) > 3\ln a_k$	**M1**	Applies result given in part **(a)**.
	$\geqslant 3^k \ln 2$	**M1 A1**	Applies inductive hypothesis and writes in required form.
	So true when $n = k+1$. By induction, true for all integers $n \geqslant 2$.	**A1**	States conclusion, all previous marks must be gained.
		6	
3(b)	$3\ln\left(a_n + \frac{1}{a_n}\right) - \ln a_n > 2\ln a_n$	**M1**	Applies results given in parts **(a)** and **(b)**.
	$> 2\times 3^{n-1}\ln 2 = 3^{n-1}\ln 4$	**A1**	AG
		2	

Q4

Question	Answer	Marks	Guidance
2	$\frac{dy}{dx} = axe^{ax} + e^{ax} = (ax+1)e^{ax}$ so true when $n = 1$.	**M1 A1**	Differentiates once using the product rule.
	Assume that $\frac{d^k y}{dx^k} = \left(a^k x + ka^{k-1}\right)e^{ax}$.	**B1**	States inductive hypothesis.
	$\frac{d^{k+1} y}{dx^{k+1}} = a\left(a^k x + ka^{k-1}\right)e^{ax} + e^{ax}\left(a^k\right) = \left(a^{k+1} + (k+1)a^k\right)e^{ax}$	**M1 A1**	Differentiates kth derivative.
	So true when $n = k+1$. By induction, true for all positive integers n.	**A1**	States conclusion.
		6	

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Q5

Question	Answer	Marks	Guidance
5	$\frac{\mathrm{d}}{\mathrm{d}x}(x\sin x) = x\cos x + \sin x = (-1)^0(x\cos x + (2(1)-1)\sin x)$	**B1**	Checks base case using product rule.
	Assume true for $n = k$, so $\frac{\mathrm{d}^{2k-1}}{\mathrm{d}x^{2k-1}}(x\sin x) = (-1)^{k-1}(x\cos x + (2k-1)\sin x)$	**B1**	States inductive hypothesis.
	Then $\frac{\mathrm{d}^{2k}}{\mathrm{d}x^{2k}}(x\sin x) = (-1)^{k-1}(-x\sin x + 2k\cos x)$	**M1 A1**	Differentiates once. Must have correct LHS for A1.
	$\frac{\mathrm{d}^{2k+1}}{\mathrm{d}x^{2k+1}}(x\sin x) = (-1)^{k-1}(-x\cos x - \sin x - 2k\sin x)$ $= (-1)^k(x\cos x + (2k+1)\sin x)$	**M1 A1**	Differentiates again.
	So, it is also true for $n = k+1$. Hence, by induction, true for all positive integers.	**A1**	States conclusion.
		7	

Q6

Question	Answer	Marks	Guidance
2	$7^2 - 1 = 48$ is divisible by 12.	**B1**	Checks base case.
	Assume that $7^{2k} - 1$ is divisible by 12 for **some** positive integer k.	**B1**	States inductive hypothesis.
	Then $7^{2k+2} - 1 = 49 \cdot 7^{2k} - 1 = 48 \cdot 7^{2k} + 7^{2k} - 1$ Or $(7^{2k+2} - 1) - (7^{2k} - 1) = 48 \cdot 7^{2k}$	**M1**	Separates $7^{2k} - 1$.
	$7^{2k+2} - 1$ is divisible by 12.	**A1**	
	So if is true for $n = k$ (could be written earlier) it is also true for $n = k+1$. Hence, by induction, true for every positive integer n.	**A1**	Depends on previous M1 and A1.
		5	

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Q7

Question	Answer	Marks
2(a)	$u_1 = 1 = 2^1 - 1$	**B1**
	Assume that it is true for $n = k$, so $u_k = 2^k - 1$.	**B1**
	Then $u_{k+1} = 2(2^k - 1) + 1 = 2^{k+1} - 1$	**M1 A1**
	So, it is also true for $n = k + 1$. Hence, by induction, true for all positive integers.	**A1**
		5
2(b)	$\frac{u_{2n}}{u_n} = \frac{2^{2n} - 1}{2^n - 1} = \frac{\left(2^n - 1\right)\left(2^n + 1\right)}{\left(2^n - 1\right)} = 2^n + 1$	**M1 A1**
		2

Q8

Question	Answer	Marks	Guidance
2	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{ax+1} = (-1)^0 \frac{0!a^1}{(ax+1)^1}$ so true for $n = 1$.	**M1 A1**	Proves base case.
	Assume that $\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = (-1)^{k-1} \frac{(k-1)!a^k}{(ax+1)^k}$ for some positive integer k.	**B1**	States inductive hypothesis.
	Then $\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = -ka(-1)^{k-1} \frac{(k-1)!a^k}{(ax+1)^{k+1}} = (-1)^k \frac{k!a^{k+1}}{(ax+1)^{k+1}}$ so true for $n = k + 1$.	**M1 A1**	Differentiates k^{th} derivative.
	By induction, true for every positive integer n.	**A1**	States conclusion.
		6	

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Q9

Question	Answer	Marks	Guidance
8(i)	$1=\dfrac{z^1-1}{z-1}$ So true when $n=1$.	**B1**	Shows base case.
	Assume that $1+z+\ldots+z^{k-1}=\dfrac{z^k-1}{z-1}$	**B1**	States inductive hypothesis.
	Then $1+z+\ldots+z^{k-1}+z^k=\dfrac{z^k-1}{z-1}+z^k=\dfrac{z^k-1+z^k(z-1)}{z-1}=\dfrac{z^{k+1}-1}{z-1}$, so true when $n=k+1$	**M1 A1**	Combines fractions.
	$H_k \rightarrow H_{k+1}$ Hence, by induction, true for all positive integers.	**A1**	States conclusion.
		5	

Q10

Question	Answer	Marks	Guidance
1	$3^3-1=26$ is divisible by 13	**B1**	Checks base case.
	Assume that $3^{3k}-1$ is divisible by 13 for some positive integer k	**B1**	States inductive hypothesis.
	Then $3^{3k+3}-1=3^3 3^{3k}-1=26\cdot 3^{3k}+3^{3k}-1$	**M1**	Separates $3^{3k}-1$
	is divisible by 13	**A1**	
	$H_k \Rightarrow H_{k+1}$ By induction, $3^{3n}-1$ is divisible by 13 for every positive integer n.	**A1**	States conclusion.
		5	

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Q11

Question	Answer	Marks	Guidance
2	We have that $f(1)=9$ is divisible by 9.	**B1**	Checks base case.
	Assume that $f(k)$ is divisible by 9.	**B1**	Makes general statement.
	$f(k+1)+f(k)=2^{3k+3}+8^{k}+2^{3k}+8^{k-1}=$	**M1**	**Uses** expansion of $f(k+1)$.
	$2^3.2^{3k}+8.8^{k-1}+2^{3k}+8^{k-1}$ OE	**A1**	Correct split of powers
	$=9\left(2^{3k}+8^{k-1}\right)$ OE so $f(k+1)$ is divisible by 9.	**A1**	Alt method: $f(k+1)=2^{3k+3}+8^{k}$
	So if f(k) is divisible by 9, so is $f(k+1)$, (and $f(1)$ is divisible by 9), $f(n)$ is divisible by 9 for every integer $n \geqslant 1$	**A1**	$=2^3.2^{3k}+8.8^{k-1}$ M1 $=8f(k)$ A1
		6	

Q12

Question	Answer	Marks	Guidance
9(i)	$P_n:\ u_n=4\left(\frac{5}{4}\right)^n+3$ Let $n=1$ then $4\left(\frac{5}{4}\right)+3=8\Rightarrow P_1$ true.	**B1**	States proposition. Proves base case.
	Assume P_k is true for some k. Then	**B1**	States inductive hypothesis.
	$u_{k+1}=\frac{1}{4}\left(5\left(4\left(\frac{5}{4}\right)^k+3\right)-3\right)=correct\ step$	**M1**	Proves inductive step.
	$=4\left(\frac{5}{4}\right)^{k+1}+3,$	**A1**	
	So $P_k \Rightarrow P_{k+1}$. Therefore, by induction, P_n is true for all positive integers.	**A1**	States conclusion.

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9(ii)	$(u_n - 3)x^n = 4x^n\left(\frac{5}{4}\right)^n = 4\left(\frac{5x}{4}\right)^n$ so r = $\left(\frac{5x}{4}\right)$	**M1**	
	So series is convergent for $-1 < \frac{5x}{4} < 1 \Rightarrow -\frac{4}{5} < x < \frac{4}{5}$	**A1**	
9(iii)	$\sum_{n=1}^{N} \ln(u_n - 3) = \sum_{n=1}^{N} \ln\left(4\left(\frac{5}{4}\right)^n\right)$ $= \left(\ln\frac{5}{4}\right)\sum_{n=1}^{N} n + \sum_{n=1}^{N} \ln 4$	**M1**	Alt method: $\sum_{n=1}^{N} \ln(u_n - 3) = ln\prod 4\left(\frac{5}{4}\right)^n$ M1 $= \ln 4^N \prod_{1}^{N}\left(\frac{5}{4}\right)^n$ $= N\ln 4 + \ln\left(\frac{5}{4}\right)^{\Sigma n}$
	$= \frac{1}{2}N(N+1)\ln\frac{5}{4} + N\ln 4$ Use $\sum_{n=1}^{N} n = \frac{1}{2}N(N+1)$.	**M1**	$= N\ln 4 + \frac{N(N+1)}{2}\ln(\frac{5}{4})$ M1
	$= N^2 \ln\frac{\sqrt{5}}{2} + N\ln\left(2\sqrt{5}\right) \Rightarrow a = \frac{\sqrt{5}}{2},\ b = 2\sqrt{5}$ oe Alt method: Writes series as an AP M1, uses summation formula M1 Correct answer A1	**A1**	$= N^2 \ln\frac{\sqrt{5}}{2} + N\ln\left(2\sqrt{5}\right) \Rightarrow a = \frac{\sqrt{5}}{2},\ b = 2\sqrt{5}$ A1
		10	

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Q13

Question	Answer	Marks	Guidance
3(i)	$u_1 < 3 (\text{given})$	**B1**	States base case.
	Assume that $u_k < 3$	**B1**	States inductive hypothesis.
	Then $3 - u_{k+1} = 3 - \dfrac{4u_k + 9}{u_k + 4} = \dfrac{-u_k + 3}{u_k + 4} > 0 \Rightarrow\ u_{k+1} < 3$	**M1 A1**	
	Hence, by induction, $u_n < 3$ for all $n \geqslant 1$.	**B1**	States conclusion.
		5	
3(ii)	$u_{n+1} - u_n = \dfrac{4u_n + 9}{u_n + 4} - u_n = \dfrac{-u_n^2 + 9}{u_n + 4}$	**M1 A1**	Considers $u_{n+1} - u_n$.
	So $u_n < 3 \Rightarrow u_{n+1} - u_n > 0$.	**B1**	Uses $u_n < 3$
		3	

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Q14

Question	Answer	Marks	Guidance
6	$y^{(1)} = e^x u^{(1)} + ue^x = e^x\left(\binom{1}{0}u + \binom{1}{1}u^{(1)}\right) \Rightarrow H_1$ is true	**M1A1**	Shows base case using product rule
	Assume that $H_k : y^{(k)} = e^x\left(\binom{k}{0}u + \binom{k}{1}u^{(1)} + \cdots + \binom{k}{r}u^{(r)} + \cdots + \binom{k}{k}u^{(k)}\right)$	**B1**	States inductive hypothesis.
	Then $y^{(k+1)} = e^x\left(\binom{k}{0}u + \binom{k}{1}u^{(1)} + \cdots + \binom{k}{r}u^{(r)} + \cdots + \binom{k}{k}u^{(k)}\right) +$	**M1**	Differentiates using product rule
	$e^x\left(\binom{k}{0}u^{(1)} + \binom{k}{1}u^{(2)} + \cdots + \binom{k}{r}u^{(r+1)} + \cdots + \binom{k}{k}u^{(k+1)}\right)$ $= e^x\left(\binom{k}{0}u + \ldots\left(\binom{k}{r} + \binom{k}{r-1}\right)u^r + \ldots\binom{k}{k}u^{(k+1)}\right)$	**M1A1**	Shows application of $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$.
	$= e^x\left(\binom{k+1}{0}u + \ldots\binom{k+1}{r}u^r + \ldots\binom{k+1}{k+1}u^{(k+1)}\right)$	**B1**	Shows reasoning for first and last term correctly
	So H_k *implies* H_{k+1} so, by induction, H_n is true for all $n \geqslant 1$.	**A1**	States conclusion.
		8	

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Q15

Question	Answer	Marks	Guidance
2	Let P_n be the proposition that 5^n+3 is divisible by 4 $5^0+3=4 \Rightarrow P_0$ is true (allow P_1)	**B1**	Some explanation of what P_k being true means
	Assume that P_k is true for some non-negative integer k.	**B1**	or e.g. $5^k+3 = 4\alpha$ for 2nd **B1**
	$5^{k+1}+3=5(4\alpha-3)+3$	**M1**	*Alt method:* Use f(k+1) – f(k) **M1 A1**
	$=20\alpha-12=4(5\alpha-3)$ (or shows that $5^{k+1}+3=5.5^k+5.3-4.3=5(5^k+3)-4.3$)	**A1**	
	P_0 is true and $P_k \Rightarrow P_{k+1}$, hence P_n is true for all non-negative integers n.	**A1**	
	Total:	**5**	

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Q16

Question	Answer	Marks	Guidance
3	When $n=1$ $\quad 1\times\ln 2=\ln 2=\ln\left(\frac{2^1}{1!}\right)\Rightarrow (H_1$ is true $)$	**B1**	
	Assume, for some positive integer k, that $\sum_{r=1}^{k} r\ln\left(\frac{r+1}{r}\right)=\ln\left(\frac{[k+1]^k}{k!}\right)$	**B1**	
	Hence $\sum_{r=1}^{k+1} r\ln\left(\frac{r+1}{r}\right)=\ln\left(\frac{(k+1)^k}{k!}\right)+(k+1)\ln\left(\frac{k+2}{k+1}\right)$	**B1**	
	$=\ln\frac{(k+1)^k(k+2)^{k+1}}{k!(k+1)^{k+1}}$	**M1**	
	$=\ln\frac{(k+1)^k(k+2)^{k+1}}{(k+1)!(k+1)^k}$ $=\ln\frac{(k+2)^{k+1}}{(k+1)!}$	**A1**	
	Thus $H_k\Rightarrow H_{k+1}$ and hence by PMI H_n is true for all positive integers.	**A1**	
	Total:	**6**	

Q17

Question	Answer	Marks	Guidance
3(i)	$\frac{d^{n+1}}{dx^{n+1}}\left(x^{n+1}\ln x\right)=\frac{d^n}{dx^n}\left(x^{n+1}.\frac{1}{x}+(n+1)x^n\ln x\right)=$ $\frac{d^n}{dx^n}\left(x^n+(n+1)x^n\ln x\right)$	**M1A1**	AG
		2	

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3(ii)	Assume H_k is true $\Rightarrow \dfrac{d^k}{dx^k}\left(x^k \ln x\right) = k!\left\{\ln x + 1 + \dfrac{1}{2} + \ldots + \dfrac{1}{k}\right\}$	**B1**	Statement of H_k seen
	$\dfrac{d^{k+1}}{dx^{k+1}}\left(x^{k+1}\ln x\right) = \dfrac{d^k}{dx^k}\left(x^k + [k+1]x^k \ln x\right)$	**M1**	
	$= k! + [k+1]k!\left\{\ln x + 1 + \dfrac{1}{2} + \ldots + \dfrac{1}{k}\right\}$	**A1**	
	$= (k+1)!\left\{\ln x + 1 + \dfrac{1}{2} + \ldots + \dfrac{1}{k+1}\right\} \Rightarrow H_{k+1}$ is true	**A1**	
	Check H_1 is true **and** H_k is true $\Rightarrow H_{k+1}$ is true; hence, by PMI, H_n is true for all positive integers n.	**A1**	
		5	

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Q18

3	For $n=1$ $10+192+5=207=9\times 23 \Rightarrow H_1$ is true. Assume H_k is true for some positive integer $k \Rightarrow 10^n+3.4^{n+2}+5=9\alpha$ Let f(n) = $10^n+3.4^{n+2}+5$ Hence $f(n+1)-f(n)=10^n(10-1)+3.4^{n+2}(4-1)$ $=9\left(10^n+4^{n+2}\right)$ $=9\beta$ Hence $f(n+1)\left(=9(\beta+\alpha)\right) \Rightarrow H_{k+1}$ is true H_1 is true and $H_k \Rightarrow H_{k+1}$, hence by PMI H_n is true for all positive integers n. N.B. Or can show $f(n+1)=9\left(10\alpha-2.4^{n+2}-5\right)$ for **M1A1A1**. (3rd ,4th&5th marks)	**B1** **B1** **M1** **A1** **A1** **A1** [6]

Q19

2	With $n=3$, $\frac{1}{2}n(n-3)=0$ A triangle has no diagonals ⇒ H_3 is true. Assume H_k is true: A k-gon has $\frac{1}{2}k(k-3)$ diagonals for some integer ≥ 3 Adding an extra vertex, a further $(k-1)$ diagonals can be drawn. $\frac{1}{2}k(k-3)+k-1=\frac{k^2-3k+2k-2}{2}=\frac{(k+1)(k-2)}{2}$ $=\frac{1}{2}(k+1)(k+1-3)$ (So $H_k \Rightarrow H_{k+1}$) ⇒H_n is true for all integers $n \geqslant 3$.	**M1** **A1** **B1** **M1** **A1** **A1** [6]

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Q20

4	$\binom{n}{r-1}+\binom{n}{r} = \frac{n!}{(r-1)!(n-r+1)!}+\frac{n!}{r!(n-r)!} = \frac{n!}{(r-1)!(n-r)!}\left(\frac{1}{n-r+1}+\frac{1}{r}\right)$ $= \frac{n!}{(r-1)!(n-r)!}\left(\frac{r+n-r+1}{r(n-r+1)}\right) = \frac{(n+1)!}{r!(n-r+1)!} = \binom{n+1}{r}$	**M1** **A1**	[2]
	$(a+x)^1 = \binom{1}{0}a+\binom{1}{1}x = a+x \Rightarrow \mathrm{H}_1$ is true.	**B1**	
	Assume H_k is true,i.e. $(a+x)^k = \binom{k}{0}a^k+\binom{k}{1}a^{k-1}x+\ldots+\binom{k}{r}a^{k-r}x^r+\ldots+\binom{k}{k}x^k$	**B1**	
	Multiplying by $(a+x)$, the coefficient of $a^{k-r+1}x^r$ is: $\binom{k}{r-1}+\binom{k}{r}=\binom{k+1}{r}$ $\Rightarrow \mathrm{H}_{k+1}$ is true.	**M1**	
	Hence H_n is true for all positive integers.	**A1**	[4]

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Q21

3	$a_1 > 5$ (given) $\Rightarrow$ H_1 is true. Assume H_k is true for some positive integer k, i.e. $a_k = 5 + \delta$, where $\delta > 0$. $a_{k+1} - 5 = \frac{4a_k^{\,2} + 25}{5a_k} - 5 = \frac{4a_k^{\,2} + 25 - 25a_k}{5a_k} = \frac{(4a_k - 5)(a_k - 5)}{5a_k} > 0 \ , \Rightarrow a_{k+1} > 5$ **Or** $a_{k+1} = \frac{4}{5}(5+\delta) + \frac{5}{5+\delta}$, $= 4 + \frac{4}{5}\delta + (1 - \frac{\delta}{5} + \frac{\delta^2}{25} - ...)$ for $0 < \delta < 5$ $= 5 + \frac{3}{5}\delta + 0(\delta^2) \geqslant a_{k+1} > 5$, ($\delta \geqslant 5$ is trivial). $H_k \Rightarrow H_{k+1}$ and H_1 is true, hence by mathematical induction, the result is true for all $n \in \mathbf{Z}^+$ (N.B. The minimum requirement is 'true for all positive integers'.) $a_{k+1} - a_k = \frac{5}{a_k} - \frac{1}{5}a_k$ $\frac{5}{a_k} < 1$ and $\frac{1}{5}a_k > 1 \Rightarrow a_{k+1} - a_k < 0 \Rightarrow a_{k+1} < a_k$	B1 B1 M1A1 (M1) (A1) A1 (5) M1 A1 (2) **Total: 7**

Q22

3	$H_k : \sum_{r=1}^{k} \frac{1}{(2r)^2 - 1} = \frac{k}{2k+1}$ is true for some integer k. $\frac{1}{2^2 - 1} = \frac{1}{3} = \frac{1}{2 \times 1 + 1} \Rightarrow H_1$ is true. $\frac{k}{2k+1} + \frac{1}{(2k+2)^2 - 1} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)}$ $= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2[k+1]+1}$ $\therefore H_k \Rightarrow H_{k+1}$ $\therefore$ (By Principle of Mathematical Induction) H_n is true for all positive integers n. (This mark requires all previous marks.) $\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1} = \frac{1}{2}$	B1 B1 M1A1 A1 A1 (6) B1 (1) **Total 7**

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Q23

3	$n = 1$ in formula gives $a^0 e^{ax} + axe^{ax} = e^{ax} + axe^{ax}$ $\frac{d}{dx}(xe^{ax}) = e^{ax} \times 1 + x.ae^{ax} = e^{ax} + axe^{ax} \Rightarrow H_1$ is true oe Assume H_k is true, i.e. $\frac{d^k}{dx^k}(xe^{ax}) = ka^{k-1}e^{ax} + a^k xe^{ax}$. $\frac{d^{k+1}}{dx^{k+1}}(xe^{ax}) = ka^k e^{ax} + a^k e^{ax} + a^{k+1}xe^{ax}$ $= (k+1)a^k e^{ax} + a^{k+1}xe^{ax}$ $\Rightarrow H_{k+1}$ is true, hence by PMI H_n is true for all positive integers n.	**B1** **B1** **B1** **M1** **A1** **A1** [6] **Total 6**

Q24

3	H_k: $f(k) = 11^{2k} + 25^k + 22 = 24\lambda$	B1
	$f(0) = 1 + 1 + 22 = 24 = 1 \times 24 \Rightarrow H_0$ is true.	B1
	$f(k+1) - f(k) = 11^{2k+2} + 25^{k+1} + 22 - (11^{2k} + 25^k + 22)$	M1
	$= 11^{2k}(121 - 1) + 25^k(25 - 1)$	A1
	$= 11^{2k} \times 24 \times 5 + 25^k \times 24 = 24\mu$	A1
	Alternatively:	
	$f(k+1) = 11^{2k+2} + 25^{k+1} + 22$	(M1)
	$= 121.11^{2k} + 25.25^k + 22 = (120 + 1)11^{2k} + (24 + 1)25^k + 22$ (OE)	(A1)
	$= 120.11^{2k} + 24.25^k + 24\lambda = 24\mu$	(A1)
	$\Rightarrow f(k+1) = 24\mu + 24\lambda = 24(\mu + \lambda) \Rightarrow H_{k+1}$ is true.	A1
	Hence by PMI H_n is true for all non-negative integers. (Must see non-negative integers.)	[6]
	CSO: Final mark requires all previous marks.	

Q25

3	$\phi(1) = 5 \times 5 - 1 = 24$ which is divisible by $8 \Rightarrow H_1$ is true.	B1
	Assume P_k is true for some positive integer $k \Rightarrow \phi(k) = 8l$	B1
	$\phi(k+1) - \phi(k) = 5^{k+1}(4k + 5) - 1 - 5^k(4k + 1) + 1$	M1
	$= 5^k(20k + 25 - 4k - 1)$	A1
	$= 5^k(16k + 24) = 8m$	A1
	$\therefore \phi(k+1) = 8(l + m)$	A1
	Hence, by PMI, true for all positive integers n. (CWO – all previous marks required.)	A1
		[7]
	Alternatively	
	$\phi(k+1) = 5^{k+1}(4k + 5) - 1$	
	$= 5.(4k.5^k) + 25.5^k - 1$	
	$= 5(8l - 5^k + 1) + 25.5^k - 1$	(M1A1)
	$= 40l + 20.5^k + 4$	
	$= 40l + 24.5^k - 4.5^k + 4$	(A1)
	$= 40l + 24.5^k - 4(5^k - 1)$	
	$= 40l + 24.5^k - 4(8l - 4k.5^k)$	
	$= 8l + 24.5^k + 16k.5^k$	(A1)
	$= 8m$	

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Q26

3	$2! - S_1 = 1$, $3! - S_2 = 1$, $4! - S_3 = 1$, $5! - S_4 = 1$ (Two correct B1, all four correct B2)	B2,1,0 (2)
	$S_n = (n+1)! - 1$	B1 (1)
	$2! - 1 = 2 - 1 = 1 \Rightarrow H_1$ is true.	B1
	H_k: $S_k = (k+1)! - 1$	B1
	$(k+1)! - 1 + (k + 1) \times (k + 1)!$	
	$= (k + 1)!(1 + k + 1) - 1$	M1
	$= ([k+1] + 1)! - 1$ Hence $H_k \Rightarrow H_{k+1}$	
	So result holds for all positive integers (by PMI).	A1 (4)
		[7]

Q27

2	Proves base case.	P_n: $5^{2n}-1$ is divisible by 8. $5^2-1=24=3\times 8 \Rightarrow P_1$ is true	B1 B1		
	States inductive hypothesis.	Assume P_k is true: $5^{2k}-1=8\lambda$ for some k.	M1		
		$5^{2k+2}-1=25.5^{2k}-1=24.5^{2k}+5^{2k}-1$ $=3\times 8.5^{2k}+8\lambda$			
	Proves inductive step.	$\therefore P_k \Rightarrow P_{k+1}$	A1		
	States conclusion.	(Since P_1 is true and $P_k \Rightarrow P_{k+1}$) $\therefore$ P_n is **for every positive integer** n (by PMI).	A1	5	**[5]**

Q28

3	Differentiates once.	$\frac{d}{dx}\left(e^x \sin x\right)=e^x \sin x+e^x \cos x$	B1		
	Rearranges.	$=\sqrt{2}e^x\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)$	M1		
	Shows true for $n=1$.	$=\sqrt{2}e^x \sin\left(x+\frac{1}{4}\pi\right) \Rightarrow H_1$ true.	A1		
	States inductive hypothesis. (May be seen by implication.)	$H_k : \frac{d^k}{dx^k}(e^x \sin x)=\left(\sqrt{2}\right)^k e^x \sin\left(x+\frac{1}{4}k\pi\right)$	B1		
	Differentiates.	$\frac{d^{k+1}}{dx^{k+1}}=\left(\sqrt{2}\right)^k\left(\sin\left(x+\frac{1}{4}k\pi\right)e^x+e^x\cos\left(x+\frac{1}{4}k\pi\right)\right)$	M1		
	Rearranges.	$=\left(\sqrt{2}\right)^{k+1}e^x\left(\frac{1}{\sqrt{2}}\sin\left(x+\frac{1}{4}k\pi\right)+\frac{1}{\sqrt{2}}\cos\left(x+\frac{1}{4}k\pi\right)\right)$	A1	7	**[7]**
	Shows $H_k \Rightarrow H_{k+1}$	$=\left(\sqrt{2}\right)^{k+1}e^x \sin\left(x+\frac{1}{4}(k+1)\pi\right) \Rightarrow H_{k+1}$ true.	A1		
	and states conclusion.	$\therefore$,by PMI, true for all positive integers n. (CWO)			

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Q29

5 (i)	Differentiates once,	$y'=2(1+x)\ln(1+x)+(1+x)$	B1		
	twice	$y''=2\ln(1+x)+3$	B1		
	and three times.	$y'''=\frac{2}{1+x}$	B1	3	
		(Allow B1 if constant term in previous line incorrect.)			
(ii)	Proves base case.	$\frac{d^3y}{dx^3}=\frac{(-1)^2.2.0!}{1+x}=\frac{2}{1+x} \Rightarrow H_3$ is true.	B1		
	States inductive hypothesis.	$H_k : \frac{d^k y}{dx^k}=\frac{(-1)^{k-1}.2.(k-3)!}{(1+x)^{k-2}}$ for some k.	B1		
	Differentiates	$\frac{d^{k+1}y}{dx^{k+1}}=(-1)^{k-1}.2(k-3)!(-1)(k-2)(1+x)^{-(k-1)}$	M1		
	Proves inductive step and	$=\frac{(-1)^k.2.(k-2)!}{(1+x)^{k-1}} \Rightarrow H_{k+1}$ is true	A1		
	states conclusion.	Hence by PMI H_n is true for all integers $\geqslant 3$	A1	5	**[8]**

Q30

9	States inductive hypothesis	H_k: $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$ for some k	B1		
	Proves $H_k \Rightarrow H_{k+1}$	$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)(\cos k\theta + i\sin k\theta)$ $= (\cos\theta\cos k\theta - \sin\theta\sin k\theta) + i(\sin\theta\cos k\theta + \cos\theta\sin k\theta)$ $= \cos[k+1]\theta + i\sin[k+1]\theta \quad \therefore H_k \Rightarrow H_{k+1}$	M1 A1 A1		
	States conclusion	H_1 is trivially true, so true for all positive integers	A1	**(5)**	
	Uses de M's Thm. to find $2i\sin n\theta$	$z^n - \frac{1}{z^n} = (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)$ $= 2i\sin n\theta$	B1		
	Uses binominal expansion	$\left(z - \frac{1}{z}\right)^5 = (2i\sin\theta)^5 = 32i\sin^5\theta$	B1		
	and groups	$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$ $= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$	M1 A1		
	Applies above result and obtains $\sin^5$	$32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$ $\sin^5\theta = \frac{1}{16}(\sin^5\theta - 5\sin 3\theta + 10\sin\theta)$	M1 A1	**(6)**	**[11]**

Q31

2	(States proposition.)	(P_n: $4^n > 2^n + 3^n$)			
	Proves base case.	Let $n = 2$, $16 > 4 + 9 \Rightarrow P_2$ is true.	B1		
	States inductive hypothesis.	Assume P_k is true $\Rightarrow 4^k > 2^k + 3^k$	B1		
	Proves inductive step.	$4^{k+1} = 4.4^k > 4(2^k + 3^k) = 4.2^k + 4.3^k$ $> 2.2^k + 3.3^k = 2^{k+1} + 3^{k+1}$ $\therefore P_k \Rightarrow P_{k+1}$	M1 A1		
	States conclusion.	Hence result true, by PMI, for all integers $n \geq 2$.	A1 (CWO)	5	**[5]**

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Q32

5	Integrates by parts to obtain reduction formula.	$\int_0^\infty x^n \mathrm{e}^{-2x}\mathrm{d}x = \left[x^n \frac{-\mathrm{e}^{-2x}}{2}\right]_0^\infty + \int_0^\infty nx^{n-1}\frac{\mathrm{e}^{-2x}}{2}\mathrm{d}x$	M1		
		$= \frac{n}{2} I_{n-1}$ (AG)	A1	2	
	(States proposition.)	$\mathrm{P}_n: I_n = \frac{n!}{2^{n+1}}$			
	Proves base case.	$n=1 \quad I_0 = \int_0^\infty \mathrm{e}^{-2x}\mathrm{d}x = \left[-\frac{1}{2}\mathrm{e}^{-2x}\right]_0^\infty = \frac{1}{2}$	B1		
		$I_1 = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4} = \frac{1!}{2^2}$ $\therefore \mathrm{P}_1$ true.	B1		
	Shows $\mathrm{P}_k \Rightarrow \mathrm{P}_{k+1}$.	$\mathrm{P}_k: I_k = \frac{k!}{2^{k+1}}$ for some integer k.	B1		
		$\therefore I_{k+1} = \frac{k+1}{2}\times\frac{k!}{2^{k+1}} = \frac{(k+1)!}{2^{k+2}}$	M1A1		
		$\therefore \mathrm{P}_k \Rightarrow \mathrm{P}_{k+1}$			
	States conclusion.	Hence by PMI P_n is true for all positive integers n.	A1	6	**[8]**

Q33

3	Proposition.	$\mathrm{H}_N: S_N = 1 - \frac{1}{(N+1)!}$			
	Proves base case.	$S_1 = \frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{2!} \Rightarrow \mathrm{H}_1$ is true.	B1		
	States inductive hypothesis.	H_k: Assume $S_k = 1 - \frac{1}{(k+1)!}$ is true.	B1		
	Proves inductive step.	$\Rightarrow S_{k+1} = 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+2)!-(k+2)+(k+1)}{(k+2)!}$	M1		
		$\Rightarrow S_{k+1} = 1 - \frac{1}{(k+2)!}$ $\therefore \mathrm{H}_k \Rightarrow \mathrm{H}_{k+1}$.	A1		
	States conclusion.	$\therefore$ (By PMI H_n is) true for all positive integers N.	A1	5	**[5]**

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Q34

2	(States proposition.)	$(\mathrm{P}_n: u_n = 4\left(\frac{3}{4}\right)^n - 2)$			
	Proves base case.	Let $n=1$ $4\times\frac{3}{4} - 2 = 3 - 2 = 1 \Rightarrow \mathrm{P}_1$ true.	B1		
	States Inductive hypothesis.	Assume P_k is true for some k.	B1		
	Proves inductive step.	$u_{k+1} = \frac{3\left\{4\left(\frac{3}{4}\right)^k - 2\right\} - 2}{4} = 4.\frac{3}{4}.\left(\frac{3}{4}\right)^k - \frac{6+2}{4}$	M1		
		$= 4.\left(\frac{3}{4}\right)^{k+1} - 2 \quad \therefore \mathrm{P}_k \Rightarrow \mathrm{P}_{k+1}$	A1		
	States conclusion.	$\therefore$ By PMI P_n is true $\forall$ positive integers.	A1	5	**[5]**